UNIVERSITY OF EDUCATION, WINNEBA

STUDENTS' CONCEPTUAL UNDERSTANDING OF THE ARITHMETIC MEAN AND THE STANDARD DEVIATION: A CASE STUDY AT THE UNIVERSITY OF EDUCATION, WINNEBA

GLORIA ARMAH

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A THESIS IN THE DEPARTMENT OF MATHEMATICS EDUCATION, FACULTY OF SCIENCE EDUCATION, SUBMITTED TO THE SCHOOL OF GRADUATE STUDIES, UNIVERSITY OF EDUCATION, WINNEBA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DOCTOR OF PHILOSOPHY (MATHEMATICS EDUCATION) DEGREE

NOVEMBER, 2017

DECLARATION

STUDENTS' DECLARATION

I, Gloria Armah, declare that this thesis, with the exception of quotations and references contained in published works which have all been identified and duly acknowledged, is entirely my own original work, and it has not been submitted, either in part or whole, for another degree elsewhere.

Signature: …………………….

Date: ……………………........

SUPERVISOR'S DECLARATION

We hereby declare that the preparation and presentation of this work was supervised in accordance with the guidelines for supervision of thesis as laid down by the University of Education, Winneba.

Prof. Samuel Asiedu-Addo (Principal Supervisor)

Signature: ……………….……..

Date: ……………………………..

Dr. Charles K. Assuah (Co-Supervisor)

Signature: ……………………..…

Date: …………………………….

Dr. Joseph I. Nyala (Co-Supervisor)

Signature: ……………………..…

Date: …………………………….

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DEDICATION

To my lovely husband, Mr. Stephen Kofi Armah, and my dearest children, Elvis and Eliada Armah who are God's gift to me.

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ABSTRACT

This study investigated undergraduate mathematics students' conceptual understanding of the arithmetic mean and the standard deviation after going through the Ghanaian Junior and Senior High School's core and elective mathematics curriculum. It also sought to determine the levels of conceptual knowledge of the arithmetic and standard deviation that these students have with respect to the Action, Process, Object and Schema (APOS) Theoretical Framework. The study employed the case study design. The purposive and convenience sampling techniques were used to select four hundred and thirty first year students from three different year batches admitted into the Department of Mathematics Education of the University of Education, Winneba. Two instruments were used to collect data on students' conception of the arithmetic mean and the standard deviation: a test and an interview schedule. The test was given to students in their first semesters on admission before lectures for the introductory statistics course begun. Nine students were randomly selected from the 2014/15 year group and interviewed. Findings revealed that undergraduate mathematics students have conceptualized the arithmetic mean as an average, and as a computational act. They exhibited an incomplete Process conception of the arithmetic mean with respect to the APOS Framework. Participants demonstrated to have no conceptual understanding of the standard deviation, revealing not even an Action conception with respect to the APOS Framework. A recommendation was made for lecturers and tutors responsible for training teachers to ensure that they teach student teachers the concepts of the arithmetic mean and standard deviation before their computations.

CHAPTER ONE

INTRODUCTION

1.0 Overview

This introductory chapter provides the justification for the study and an overview of the entire research. It starts with a discussion of the background to the study, which is followed by the statement of the problem. The purpose, as well as the objectives of the study is clearly outlined, from which the research questions that guided the study are also enumerated. The significance of the study, delimitations of the study and finally, the organization of the thesis are also discussed.

1.1 Background to the Study

Education is the process of transmitting knowledge and values to an individual to make him socially, morally, mentally, spiritually and psychologically fit in order to be useful to himself and to the society as a whole. It takes place either formally in an organized or structured setting, or informally anywhere in the society. From the point of view of Olawoye and Salman (2008), education is the most veritable means of social growth; a formidable force, a dependable and an indispensable tool for a nation's development.

According to Salman, Yahaya and Adewara (2011), in the design of education, citizens are equipped with high level skills needed for the development of a nation. It transforms and builds the capacity for individuals to acquire suitable information, skills and proficiency for one's own survival and the development of the society (Egbochuku & Alika, 2008). They stressed that education closes the door to poverty and ignorance, and also opens the door to economic, social and political developments.

In the school setting, these aims and objectives of education as described by Egbochuku and Alika (2008), can be achieved through the subjects teachers teach, including mathematics. Mathematics is the queen of science, the language of nature, and the bedrock of national development, a subject without which a nation cannot advance scientifically and technologically (Alutu & Eraikhuemen, 2004). Sells (1980) identified mathematics as the critical filter that restricts people from gaining access into a lot of high-status and high-income careers by acting as an open door to many careers and fields of study. It is a basic course for scientific and technological development of any country (Salman, Yahaya & Adewara, 2011), a part of life without which man cannot function (Nabie, 2002).

Today's world requires young people to use numbers proficiently, read and explain numerical data, reason logically, solve problems, as well as communicate effectively with others using precise mathematical data and interpretations. This requires knowledge in mathematics (CRDD, 2012b). As a subject, mathematics has found its way in all facets of human endeavor. It is required in learning other science subjects. As a result, students who are not firmly grounded in mathematics always have problems utilizing mathematical concepts, principles and skills in the course of their education. Due to its usefulness in buying and selling, in keeping records, in demonstrating understanding and appreciating of nature, in thinking critically and reasoning logically, mathematics has the ability in sustaining the interest of students in the formal school system (Eraikhuemen, 2001).

Countries all over the world, including Ghana, recognize the importance of mathematics and statistics in their everyday lives, hence, they include them in their school curricula. In Ghana, mathematics is one of the core subjects taken right from the lowest level of the education ladder to the tertiary levels. Junior and senior high students write mathematics examinations to determine their progression from one level to the other. It is a requirement that before a student gains admission into any tertiary institution, the student must have at least a credit pass in mathematics. In Nigeria, the same qualification is needed for admission into science, business and engineering courses in their tertiary institutions (Alutu & Eraikhuemen, 2004).

As a result of its usefulness, Ashby (2009) observed that, for years, mathematicians have been highly respected amongst their academic peers. Yet, though valued, the subject, mathematics continues to be feared by a large number of people. Studies have consistently shown that students see the subject as difficult (Awanta, 2009; Appiahene, Opoku, Akweittey, Adoba, & Kwarteng, 2014; Mutodi & Ngirande, 2014). Other studies have indicated students' dislike and fear of mathematics, irrespective of cultural and economic background, (Burns, 1998; Fredua-Kwarteng, 2004; Zaslavsky, 1994) and these results in negative experiences with mathematics as early as elementary school (Jackson & Leffingwell, 1999). Students carry along these negative experiences as they progress to the higher levels resulting in poor performance in the subject. According to Awanta (2009), the general belief is that, mathematics is the most difficult as well as the most feared subject among all school subjects, especially by female students.

Even though a lot of incentives have been put in place in Ghana to encourage and motivate more students to take mathematics seriously, performance in the subject has not been encouraging (Eshun, 1999; Mereku, 2003). The Science, Technology and Mathematics Education (STME) Clinic for girls, especially, was established in 1987. This is a two-week clinic organized yearly for selected girls studying science and mathematics to stimulate their interest in science, technology and mathematics education and also, help them to come into contact with women career scientists and technologists, and to be given career guidance on job opportunities. The ultimate aim of the STME Clinic is to make science, mathematics and technology attractive to the girls and encourage them to offer (Sutherland-Addy, 2002). There is also the Ministry of Environment, Science and Technology's scholarship scheme, Mathematics, Science and Technology Scholarship Scheme (MASTESS). The scheme, which started on 1st September 2009, aims at providing a need based financial support to deserving students studying Mathematics, Science and Technology subjects in second cycle and tertiary institutions in Ghana (Mathematics, Science and Technology Scholarships, 2013). These programmes also aim at increasing enrollment of students studying mathematics, science and technology in the various institutions as part of national effort to enhance the study of science and mathematics.

According to Ghosh (1997), new values and competencies are necessary for survival and prosperity in this rapidly changing world where technological innovations have made many skills redundant. To become abreast of current development and to acquire new knowledge, the Lisbon European Council of March 2000 placed the development of a knowledge–based society at the top of the Union's policy agenda, considering it to be the key to the long term competitiveness and personal aspirations of its citizens. Statistics education, they observed, had a crucial role to play in this.

As a discipline, statistics presents a rigorous scientific method for gaining insight into data. It is used to make sense of data to enhance decision–making (Abu

Kassim, Zatul-Iffa, Mahmud & Zainol, 2010). According to Brown and Porter (1996), statistics provides the theory and methodology for the analysis of wide varieties of data and it is indispensable in medicine, for medical studies on diseases and new drugs. Since statistics is utilized widely in many disciplines and in everyday life, a number of university departments require their graduate and even undergraduate students to take at least an introductory course in basic statistical methods (Bangdiwala, 1989).

Tishkovskaya and Lancaster (2012) have observed an increasing attention given to the teaching and learning aspects of statistics education over the last twenty years. It is widely recognized that statistics is one of the most important quantitative subjects in a university's curriculum (Watson, 1997). Teaching statistical courses is known to be challenging because the courses appeal to students from different backgrounds and competencies, many of whom have had discouraging experiences with statistics and mathematics (Garfield, 1995). Perhaps, more critical is the fact that these courses affect lifelong perceptions of attitude towards the value of statistics for many students, who are the future employees, employers and citizens (Tishkovskaya & Lancaster, 2012).

Introductory statistics courses aim at training students to be efficient users, consumers and communicators of statistics (Qian, 2011). Additionally, the pedagogy of statistics is developing to help form a statistically literate society (Kettenring, Lindsay & Siegmund, 2003). Nevertheless, sound statistical reasoning skills are not attained in one education statistics course, but rather need to be cultivated and developed over one's entire educational experience (Green & Blankenship, 2013). A lot of instructors consider statistics as an area of applied mathematics within which we can easily find real–world applications. This applied mathematics integrates probability and sampling techniques to describe and based on experience, predict outcomes. It could be argued that statistics is a necessity for an informed educated citizenry today to give a precise interpretation of statistics. This is because statistical data, summaries, and inferences are regularly encountered in the everyday activities of people than any other form of mathematical information (Abu Kassim et al., 2010).

Despite the importance of the subject, Abu Kassim et al., (2010) pointed out that statistics in all its complexities may create bias and misuse in many different ways. Citizens must therefore be aware that there can be manipulation and deception through data misrepresentation, such as repetition of studies till obtaining desired results or by using small and/or biased samples. Thus, it is important for students who are the future adult generation to be statistically literate so that they can analyze critically any information they receive (Abu Kassim et al., 2010).

In recent years there have been frequent calls by statisticians and statistical educators against the teaching and learning of statistics with emphasis on only computational ability without understanding the statistical knowledge and concepts behind it. Gal and Garfield (1997) regarding the call for reform in the teaching and learning of statistics listed eight instructional goals for statistics education to assist students comprehend and use statistical information and data in this information age. The first of the goals was to assist students to understand the big ideas that form the basis of statistical inquiry.

It was because of this call for reform that the Guidelines for Assessment and Instruction in Statistics Education (GAISE) Project was funded by the American Statistical Association in 2003 to come up with endorsed guidelines for statistical assessment and teaching in the K–12 curriculum and also for the introductory college statistics course. In this project, six recommendations were made for statistics education. These included: emphasizing statistical literacy and developing statistical thinking; using real data in teaching statistics; and stressing conceptual understanding rather than mere knowledge of procedures.

In Ghana, one of the objectives of the mathematics curriculum at the primary level is to help pupils to be able to collect, analyze and interpret data and find probability of events. So right from primary one, pupils start learning to collect, record and represent data visually. In primary five and six pupils are taught other means of representing data like the stem and leaf plot and bar graph, and also learn to read and interpret graphs. They are also introduced to finding the mean, median and mode, which are measures of central tendency (CRDD, 2012a).

At both the primary and Junior High School (JHS) levels, statistics is taught as collecting and handling data, which students classified as one of the easiest topics in mathematics (Awanta, 2009). Students at this level are taught how to collect data from simple surveys and organize them into tables, find the arithmetic mean, median and mode and determine which one suits a given situation. They are also taught to use various representations for a given data and also read and interpret data presented in tables. All these are efforts to help achieve the main rationale for the mathematics curriculum at the JHS level which is to provide knowledge and mathematical skills to pupils from various backgrounds and levels of ability to help them meet today's world demands of using numbers competently, reading and interpreting numeral data, reasoning logically, solving problems involving calculations and mathematical reasoning, as well as interacting effectively with others using accurate mathematical data and interpretations. These skills, when acquired "will help them in their chosen careers later in life and in the process benefit the society and the nation. The Ghanaian

JHS curriculum therefore emphasizes mathematical knowledge and skills that will help the Ghanaian young person to develop basic numeracy competence to be able to function effectively in society (CRDD, 2012b).

At the senior high school (SHS) level, students learn statistics in the first and second years. They are expected to construct frequency distribution tables, represent the information on suitable diagrams, interpret and draw simple inferences from tables and graphs and find the arithmetic mean. Students are also expected to construct and interpret cumulative frequency curves and use them to estimate the quartiles, deciles and percentiles. They are also to calculate and interpret the variance and standard deviation of a data. By the end of senior high in Ghana, the students are expected, as part of the general objectives of the core mathematics, to be able to organize, interpret and present information accurately in written, graphical and diagrammatic forms (CRDD, 2010). All these aim at enabling all young Ghanaian persons to acquire the mathematical skills, insights, attitudes and values that they will need to be successful in their chosen careers and daily lives. At the SHS level, students are expected to develop their expected mathematical competence so as to use the knowledge gained in solving real life problems and also to be well equipped for further studies and related vocations in mathematics, science, commerce, industry and other diverse professions (CRDD, 2010). Such training, according to Qian (2011), helps at training students to fit into this information age in order to be efficient users, consumers and communicators of statistics.

As in the case in the past, most people today still believe that mathematics is all about calculation. However, for mathematicians, calculation is merely a tool for understanding structures, relationships and patterns of concepts in mathematics, and therefore generating solutions to real life problems that are challenging. With the rapid developments and innovations in information and communication technologies, this view held by mathematicians has gained more attention and importance. It has become a necessity for everyone to reach, analyze, and apply the knowledge gained in mathematics effectively and efficiently so as to be successful citizens in this information age. Students, in particular, need to be well–prepared with advanced mathematical knowledge (Saritas & Akdemir, 2009).

1.2 Statement of the Problem

The arithmetic mean and the standard deviation are two fundamental statistical concepts taught and encountered in every introductory statistics course, because they seem to appear in numerous everyday contexts and for their predominant use in most inferential statistics. Marnich (2008) observed that, an area of statistical study in which better understanding of students' knowledge could lead to innovations in the pedagogical process is the arithmetic mean. The role it plays in statistics makes it to be used (or misused) widely in many academic and professional disciplines, and in everyday life. Though it is linked to a simple standard mathematical algorithm, students do not find it easy to understand. This ultimately leads to the misconceptions and lack of its conceptual understanding (Marnich, 2008).

Most children become aware of the vocabulary, arithmetic mean, even before they are introduced to the concept in school. They become familiar with phrases such as average height, average age, and average score, often used by their teachers and in text books before understanding its meaning (Chatzivasileiou, Michalis, and Tsaliki, 2010). Thus, learning about the arithmetic mean becomes the first time pupils encounter a number that expresses a relationship among particular numbers (Khalil, 2010).

Apart from being a representative or typical value of a set of data, the arithmetic mean is used in statistics beyond the suggestion of central tendency. It is used, for example, in calculating other statistics such as the standard deviation, creating formulas for distributions such as the Poisson and Normal, finding confidence intervals, and testing hypotheses (Marnich, 2008). The arithmetic mean can also inform or model concepts outside of statistics. Physically, it can be assumed as a center of gravity. We can think of the standard deviation as the average distance of data points from the arithmetic mean of the data set (Marnich, 2008). The ability to mathematically model concepts in physics, such as moment of inertia, to concepts in statistics signifies the potential significance of comprehensively understanding all aspects of the arithmetic mean. The arithmetic mean is a numerical measure used to describe and make meaning of a data set. With the standard deviation, it is also used as a tool to summarize and compare data sets (Cai, 1998).

The arithmetic mean, as a concept, may seem so straightforward, basic and ubiquitous, that difficulties students have with problems involving this concept must be due to lack of attention or motivation. Teachers and textbook publishersrepeatedly assume that it is simply another use of division. As a result, if children understand the concept of a fair or equal share, then they must also show an understanding of the concept of average (Mokros & Russell, 1995). This is not the case since experience with non-mathematically oriented students suggests that they often possess no more than minimal instrumental understanding (Skemp, 1976) of even the most elementary quantitative concepts. Instrumental understanding entails the recognition of a task as one whose particular rule is known by an individual. It should be emphasized that almost all students know the computational algorithm by which the arithmetic mean is calculated: the "add them up and divide formula" (Pollatsek, Lima & Well, 1981).

Research has, however, shown that children and adults alike do not have a deeper understanding of these basic concepts in statistics even after many years of schooling (Hawkins, Jolliffe & Glickman, 1992; Mathews & Clark, 2007; Armah & Asiedu-Addo, 2014). Though people are able to calculate the arithmetic mean for use in simple situations, they are not able to use it in complex contexts. Pollatsek, Lima and Well (1981) referred to students' use of the arithmetic mean as a "computational rather than a conceptual act" (p. 191). According to them, "knowledge of the mean seems to begin and end with an impoverished computational formula" (p. 191). If an individual knows how the arithmetic mean is computed, it does not automatically mean he/she understands it (Batanero, Godino, Vallecillos, Green, & Holmes, 1994; Armah & Asiedu-Addo, 2014), though it might be simple and straightforward.

In a study by Cazorla (as cited in Guimarães, Gitirana, Marques & dos Anjos, 2010), Brazilian undergraduate students were found to be able to compute the arithmetic mean of a data set, compute a new mean when a new data value is added to the set of data and also reverse the computation process to find an unknown value when given the arithmetic mean and other values. However, they did not demonstrate an understanding of the arithmetic mean as a representative value of a data set. Cai (1995) found a similar result when he also observed that though a majority of sixth-graders used in a study were conversant with the algorithm for finding the arithmetic mean, only a few demonstrated to have a conceptual understanding of it. On the other hand, in Stella's study of Brazilian high school students' abilities to use the arithmetic mean, they were rather found to have difficulties in using the algorithm for both the arithmetic mean and pondered (weighted) mean (Stella as cited in Guimarães, Gitirana, Marques & dos Anjos, 2010).

On the arithmetic mean and open-ended test items, Garcia Cruz and Garrett (2006), noted that though final year students in the secondary school chose the correct options for multiple-choice items, they totally could not show any reasonable method or conceptual understanding in solving related open ended items. They were found to be using incorrect arguments for the open ended items requiring the use of the same concept in the multiple-choice items. This meant that answers to the multiple-choice test items were chosen without any rational criteria.

The concept of variability or dispersion, like the standard deviation of a set of data is another basic concept in any introductory statistics course. It is acknowledged by statisticians and statistics educators as the core of statistics. “It is at the heart of statistics and is the fundamental component of statistical thinking” (Garfield & Ben-Zvi, 2005, p. 92). Gould (2004) noted that dispersion or variability makes statistics so challenging and fascinating and helps us to explain, model and make inferences from data.

Before an introductory statistics course, Dubreil-Frémont, Chevallier-Gaté, and Zendrera (2014) conducted a study to examine the knowledge undergraduate students had on the concept of the arithmetic mean and the standard deviation as well as examine their progress after the course, so as to evaluate their way of teaching. The results revealed that before the course, less than 20.0% of the students could give good definitions of the standard deviation (as a measure of dispersion and the square root of the variance). More than half of the university students could not define, let alone explain what the standard deviation represented. About a quarter defined it as the range and half of the participants gave other definitions that had no relation to the concept of the standard deviation. Though the percentage of those who could define it as a measure of dispersion or the square root of the variance increased after the course, still about a third of them gave definitions that had no bearing on the concept. When compared with the responses given to the arithmetic mean, results indicated that students had difficulties in understanding the concept of the standard deviation. That is to say, the students had a better conceptual understanding of the arithmetic mean than the standard deviation when tested.

In a related study, Chan and Ismail (2013) identified misconceptions in reasoning about variability among Malaysian high school students. They observed the emergence of two new misconceptions. That is: “value of standard deviation is equivalent to value of arithmetic mean”, and “same frequency equals same standard deviation”. The majority of the students showed these misconceptions, these add to the list of misconceptions students show in their reasoning about the standard deviation as a measure of dispersion. In another study by Chaphalkar and Leary (2014), they observed students giving inconsistent explanations of answers to questions requiring them to identify histograms with highest or lowest variability.

In a study by Allwood and Montgomery (1982), they observed that students in statistics courses failed to establish a conceptual basis for their solution strategies. They were also found to be describing their statistical solutions rather than justifying them. According to Garfield (1995), though students may sometimes be able to give correct answers to some test items, they may still not understand the basic ideas and concepts in statistics. Sometimes even those who understand these concepts and can as well do computations on these concepts often fail to make use of their conceptual knowledge (Trumpower, 2004).

The researcher who is a lecturer in the UEW, has taught introductory statistics to first year students for about a decade. This course is taken in their very first semester of the first year. Topics treated include: the nature and types of data, organizing data into frequency distribution tables, constructing graphical representations of data, measures of central tendency, measures of variation and the concept of probability. All these topics are introduced to students from the JHS level through to the SHS level, either in core or elective mathematics. However, the observation over the years has been that students are always able to do calculations under these topics correctly, but finds it difficult explaining the concepts underlying their calculations.

“Much research has been done that indicates that students are not learning what we want them to” (Garfield, 1995. p. 27). An observation by Aquilonius and Brenner (2005) was that students sometimes say or do things in their classes that make their instructors believe they have a better understanding of what is being taught. At other times these same students make mistakes on tests and homework that make the instructor doubt they really understand what has been taught.

Broers (2009) noted that meaningful knowledge of statistics involves more than simple factual or procedural knowledge. Thus, to be able to use statistics intelligently, conceptual understanding of the theory underlying it is important. Broers (2009) defined conceptual understanding as the “ability to perceive links and connections between important concepts that may be hierarchically organized” (p. 1). To him, researchers often refer to this type of knowledge as structural knowledge.

Some studies which have investigated students' conception of the arithmetic mean and standard deviation employed the Action, Process, Object, and Schema

(APOS) Theoretical Framework by Asiala et al., (1997) which is a framework for research and curriculum development (Mathews & Clark, 2007; Clark, Kraut, Mathews, & Wimbish, 2007). It consists of three components, namely: the theoretical analysis of a concept, the design and implementation of instruction, and observations and assessment. Its specific goals are:

- to increase our understanding of how learning mathematics can take place,
- to develop a theory-based pedagogy for use in undergraduate mathematics instruction, and
- to develop a base of information and assessment techniques which shed light on the epistemology and pedagogy associated with particular concepts (p. 4).

The first component, the theoretical analysis of a concept, which is the focus of this study aims at describing an individuals' development in understanding a mathematical concept. According to the theory, when a student possesses knowledge of a problem situation, then he/she has the ability to make mental constructions that can be used in dealing with the problem situation (Asiala et al., 1997). These mental constructions are actions, processes, objects and schemas.

The theory asserts that an individual understands a mathematical concept when he/she is able to manipulate mental or physical objects already built to form actions which may be interiorizedinto processes. These processes are then encapsulated into an object to be acted upon by other actions and processes into schemas (Asiala et al., 1997). So in analyzing the knowledge an individual possesses on a mathematical concept, he/she can be found to either have an action, process, object or schema conception of the concept.

According to Asiala et al., (1997), an individual has an action conception of a mathematical concept when he/she can only carry out operations (actions) on the concept when given rules or formulas to follows. He/she is said to be responding to external stimuli. However, when the action is repeated on several occasions and the person reflects on it, it is interiorized into a process. This means the individual makes some internal constructions relating to the action. He/she is thus able to reflect on how and why the actions work the way they do, and that helps the individual to come out with their main features, have control over them and also use them without any difficulty (Bansilal, 2014). This 'interiorization' allows the individual to be mindful of the action, so as to respond to it and combine it with other actions (Dubinsky, 1991). When that happens, then the individual is said to have a process conception of the mathematical concept. At this stage, the individual can reflect or think about the end result of the action without necessarily having to do it, and can also imagine its reversal (Arnawa, Sumarno, Kartasasmita, & Baskoro, 2007).

An individual with an object conception of a mathematical concept will be able to think of actions or operations that are applied to a process, can imagine it in its totality and also become aware that transformations which are also actions and processes can act on it, and he/she is able to construct those transformations. The individual is then said to have encapsulated the process into an object (Asiala et al., 1997). "Through encapsulation, abstract notions are conceived as objects which have properties and various representations" (Font, Trigueros, Badillo & Rubio, 2012, p. 3). However, in the course of performing actions or processes on an object, it is mostly useful that the individual is able to de-encapsulatethe object back to the process in order to focus on its fundamental properties (Asiala et al., 1997).

According to Debunsky (1991),

> A schema is a more or less coherent collection of objects and processes. A subject's tendency to invoke a schema in order to understand, deal with, organize, or make sense out of a perceived problem situation is her or his knowledge of an individual concept in mathematics. Thus an individual will have a vast array of schemas. There will be schemas for situations involving number, arithmetic, set formation, function, proposition, quantification, proof by induction, and so on throughout all of the subject's mathematical knowledge. Obviously, these schemas must be interrelated in a large, complex organization (pp. 101-102).

As a result, when a person is able to collect processes and objects on a mathematical concept and organize them into a structured manner, he/she is said to have a schema of the concept. These constructed schemas, then become objects in the construction or organization of 'higher level' schemas. By this, we say the schema has been thematized into an object (Asiala et al., 1997).

From the literature, it is an undeniable fact that students experience challenges in their bid to understand the concepts of the arithmetic mean as a measure of central tendency and the standard deviation as a measure of variation (Yolcu & Haser, 2013; delMas & Liu, 2005; Ben-Zvi & Garfield, 2004b; Chan & Ismail, 2013). However, since statistics is not taught in Ghanaian pre-tertiary schools as a separate subject or course of study, scientific research into students' performance in statistics is rare even though the general performance in mathematics is known to be low (Eshun, 1999; Mereku, 2003). Therefore the question which the present study sought to answer was: "what conceptual understanding do first year mathematics students of UEW have about the arithmetic mean as a measure of central tendency and the standard deviation as a measure of variation at their entry stage"?

1.3 The Purpose of the Study

The purpose of this study was twofold. First, it sought to find out the conceptual understanding first year mathematics students of the UEW had about the arithmetic mean as a measure of central tendency and the standard deviation as a measure of variation at their entry stage. Secondly, it sought to investigate the level of conceptual knowledge these students had of the arithmetic mean and the standard deviation with respect to the APOS Theory.

1.4 Objectives of the Study

This study was intended to:

- investigate the conceptual understanding first year mathematics students had on the arithmetic mean at their entry stage into the university.
- determine students' level of conceptual knowledge of the arithmetic mean, with respect to the APOS Theory.
- investigate the conceptual understanding first year mathematics students had on the standard deviation at their entry stage into the university.
- determine students' level of conceptual knowledge of the standard deviation, with respect to the APOS Theory.

1.5 Research Questions

The following research questions which emanated from the objectives above guided the study:

1. What conceptual understanding of the arithmetic mean do first year undergraduate mathematics students have as a measure of central tendency?

2. What level of conceptual knowledge with respect to the APOS Theory, do first year undergraduate mathematics students have about the arithmetic mean?

3. What conceptual understanding of the standard deviation do first year undergraduate mathematics students have as a measure of variation?

4. What level of conceptual knowledge with respect to the APOS Theory, do first year undergraduate mathematics students have about the standard deviation?

1.6 Significance of the Study

The aim of any good educational system is to inculcate into learners values, which would enable them to become useful to society. Since society is dynamic, the development of new strategies and techniques by educators has greatly improved teaching methods. Mathematics teachers and educators have been inspired by these desires to carry out research work in various aspects of the subject to improve teaching and learning.

The researcher believed that this study would bring to light how first year mathematics students have conceptualized the concepts of the arithmetic mean and the standard deviation. This will help enhance the efforts being made by educationstakeholders to help improve upon the problem of poor performance of our students in mathematics at the basic, JHS and the SHS levels. The findings, it is hoped, will encourage mathematics teachers to modify their teaching style to meet current trends for students to benefit from what they are taught and end up becoming useful to themselves and to the nation as a whole. Teachers will also become confident and self-motivated in their teaching when they understand the concepts they are presenting to students.

It was further anticipated that students will benefit from this study since the findings will help them redirect their learning to be able to fit into the information age and make meaning of whatever information they come into contact with. Students willagain develop interest and be more courageous in taking statistical courses or programs. Most importantly, the results of the study were projected to contribute immensely to knowledge and up-date those that have been done earlier and serve as resource material for all stakeholders and others who would like to research further into this area of national interest.

1.7 Delimitations of the Study

Even though the study focused on students admitted into the department of mathematics education of the UEW, only those who studied elective mathematics were considered for the study. The researcher believed that those who did not study elective mathematics might have not delved deeper into the study of these concepts under study as those who did at the SHS level.

1.8 Limitations of the study

The results of this research study could not be generalized due to the following reasons:

- Only students from the Department of Mathematics Education were included in the study. Students from other departments and even other universities who studied elective mathematics at the SHS could have been added to give a broader view of the situation at hand. However, students who attended SHS from all the ten regions of Ghana were represented.
- The non-availability of information on students' performance in statistics in the Ghanaian situation was another limitation. This is because statistics is not taught as a subject at the pre-tertiary level but rather as a sub topic in mathematics. As such, it was difficult getting studies on students' performance in statistics to compare with.

1.9 Organization of the Thesis

This thesis is structured into five chapters. Chapter One is responsible for the motivation or inspiration for the research. This includes a discussion on the relationships between mathematics and statistics, the purpose, objectives as well as the significance of the study. Chapter Two reviews the relevant literature to help in developing knowledge on the subject of the arithmetic mean and the standard deviation. It looks at some general views on teaching for understanding or conceptual knowledge and understanding of statistical concepts and how they relate to statistical literacy, reasoning and thinking. The APOS theoretical framework, the main theoretical framework for the study is discussed.

Chapter Three is about the research methodology. This includes an account of the population, instruments, and the entire procedure of data collection and analysis. Issues of trustworthiness of the study as well as the subjectivity statement of the researcher are also considered in chapter three. The demographic characteristics of the sample, results and the findings of the analyzed data are identified, interpreted and discussed in Chapter Four. This will highlight the major findings and the inferences made from them in relation to findings from related previous studies. Chapter Five, the final chapter, looks at the summary of the research study, summary of major findings, conclusions, recommendations and areas of future study based on the findings of the research.

CHAPTER TWO

LITERATURE REVIEW

2.0 Overview

This chapter discusses the review of related literature that buttresses the thesis. The following are discussed: the APOS theoretical framework, statistics educational reforms, the need for statistical literacy, reasoning and thinking, teaching for understanding, mathematical knowledge, school mathematics and its teaching in Ghana, and the concepts of the arithmetic mean, and variation.

2.1 The APOS Theoretical Framework

The main theoretical framework of this study is the APOS framework which uses qualitative research methods and has its basis from definite theoretical perspective being developed to understand Piaget's reflective abstraction and restructure them in the setting of undergraduate mathematics. With three components as illustrated in Fig 2.1, the framework starts with an initial theoretical analysis of what it means for an individual to understand a mathematical concept, and how a learner can construct that understanding. This leads to the planning of an instructional treatment that centers directly on getting students to make constructions required by the analysis and its implementation. This also leads to the final component, observation and assessment which involves data gathering. The data are then analyzed in the context of the theoretical analysis and the researchers then go through a cycle through the components to refine the theory and the instructional treatment as needed (Asiala et al., 1997). This is shown in Figure 2.1.

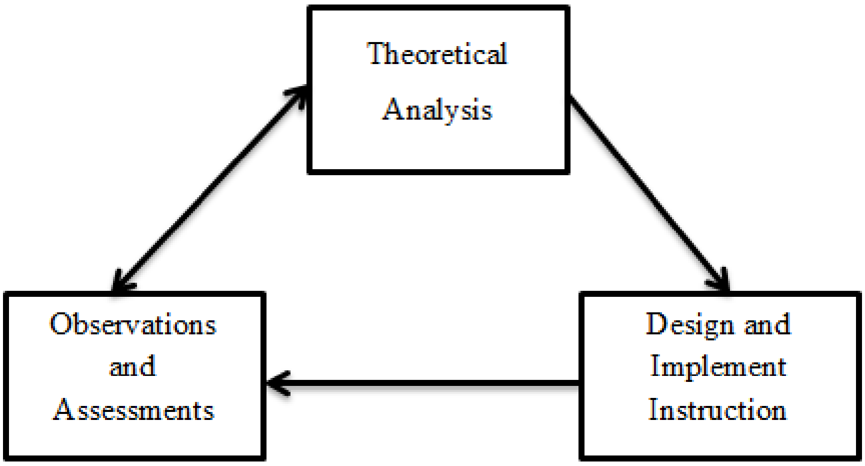


Figure 2. 1 Components of the APOS Theoretical Framework

This study dwells mainly on the theoretical analysis component of the framework, and one of the issues raised at this level is "to what extent can a theoretical analysis provide an accurate or even an approximate picture of what is going on in the minds of the learners" (p. 5). As a result, the purpose of the theoretical analysis of a mathematical concept is to suggest a model that describes the exact mental constructions made by a learner to lead to his/her understanding of the concept. The end result of this, they referred to as genetic composition of the concept (Asiala et al., 1997).

Since the theoretical perspective of the framework was on what it means to learn and know something in mathematics, Asiala et al., (1997) noted that:

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reacting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (p. 5).

They affirm that:

> Possessing" knowledge consist in a tendency to make mental constructions that are used in dealing with a problem situation. Often the construction amounts to reconstructing (or remembering) something previously built so as to repeat a previous method. But progress in the development of mathematical knowledge comes from making a reconstruction in a situation similar to, but different in important ways from, a problem previously dealt with. Then the reconstruction is not exactly the same as what existed previously, and may in fact contain one or more advances to a more sophisticated level. (p. 6)

The whole notion, they said, relates to the famous Piagetian dichotomy of assimilation and accommodation (Asiala et al., 1997). Piaget believed that a child at the sensorimotor stage of development learns through adoption which had two aspects, assimilation and accommodation. Assimilation, Piaget described as the process where a child takes in new information and incorporates it into his existing knowledge base. According to Piaget, accommodation occurs when a new information or experience cannot be assimilated thereby causing a child to alter or completely change his knowledge base, which he called schemata (Campbell, 2006; Dimitriadis & Kamberelis, 2006; Flavell, 1963). In a nutshell, what Asiala et al., (1997) are describing is in itself the outcome of reconstructing our understanding of Piaget's theory and extending it to post-secondary mathematics.

The theory describes the four specific cognitive constructions used to analyze an individual's cognitive construction of a mathematical knowledge. Therefore the theory postulates that "understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas" (p. 6). This is illustrated in Figure 2.2.

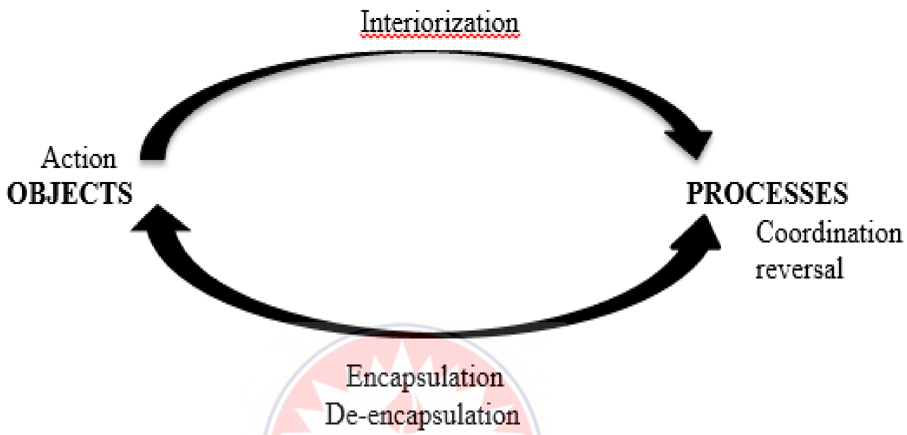


Figure 2. 2 Construction for Mathematical Knowledge

According to Hatfield (2013), "given that a construct is an idea in the mind of an observer about how an individual understands a concept, a catalytic construct serves as an observer's model of how an individual's understanding might advance within a given framework" (Hatfield, 2013, p. 1). Hatfield identified three catalytic constructs: interiorization, condensation, and reification within the APOS framework. Sfard (1992) described these catalytic constructs. He described interiorization as the mental process in which a learner becomes familiar with the processes that eventually lead to a new concept and condensation as compacting lengthy sequences of the operations used in interiorization into more manageable units. Reification, which is the last catalytic construct, Sfard described it as a qualitative leap in the thinking of an individual.

Asiala et al., (1997), gave a detailed description of each of the aforementioned mental constructions in the APOS framework:

2.1.1 Action

A transformation is considered an action, when it is a reaction to stimuli which an individual perceives as somewhat external (Maharaj, 2010) and requiring step-by-step instructions, either explicitly or from memory, on how the operation is performed (Dubinsky & McDonald, 2001). That is, if a students' understanding of a concept is limited to action conception, then that student can only operate within the concept when given external clues or steps as to what to do (Asiala et al., 1997). This type of conception, according to Thompson (1994), is like a recipe to be applied to numbers. Students with an action concept of function think that the recipe remains the same across numbers, except that they must actually be applied to some number before producing anything. They do not actually see the recipe as representing a result of its application.

For example, when a student cannot interpret a situation as a function, but can only evaluate the value of the function at a point, when given the function and the point, he is said to have an action conception of a function. With such a student, the domain, range, inverse of a function, as well as the notion that 'the derivative of a function is a function' are all difficult areas to understand. This is as a result of his inability to go past an action conception of a function (Asiala et al., 1997). Thus, if a student is able to solve a given equation using as a guide the steps in the solution of a similar equation, he is performing an action. We therefore say an individual has an action conceptionof a given concept if his depth of understanding is limited to performing actions relative to that concept (Mathews & Clark, 2007). Though an action conception is limited, this describes how actions form the vital beginning of learning a concept.

2.1.2 Process

When an action is performed on many occasions and the individual reflects on it, he then builds an internal construction that does the same action without following external clues again. The individual can visualize performing the action without actually doing it explicitly and think of or imaging reversing it and constructing it with different procedures (Dubinsky & McDonald, 2001). According to Thompson (1994), a student is said to have a process conception of function when he or she is able to build an image of "self-evaluating" expressions. Such a student has no difficulty imagining evaluating an expression to think of its results. From the perspective of such a student, an expression stands for what would be obtained when it is evaluated.

Process is seen as internal, whereas an action is seen as external (Mathews & Clark, 2007). According to Hatfield (2013), at the process stage the individual begins to condense the processes involved in actions. When this happens, he is said to have interiorized the action as a process. So "an individual with a process conceptionof a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing them" (Asiala et al., 1997, p. 7). For example, with a process conception of functions, an individual can link two or more processes to construct a composition or reverse the process to obtain inverse functions.

2.1.3 Object

According to Asiala et al. (1997), when an individual is able to think and recollect on operations carried out on a particular process of a concept, and becomes aware of this process as a whole and in addition comes to the realization that this entity possesses some characteristics, and that transformations which may either be actions or processes can act on it, and he is able to construct these transformations, then he is thinking of the process as an object. We say that the individual has encapsulated the process into an object, and as a result, he has an object conceptionof the concept.

Asiala et al. (1997) continued that in the process of performing an action or a process on an object, it is important to de-encapsulatethe object back to the process from which it came so as to use its characteristics to manipulate it. It is easy to see how encapsulating processes to objects and de-encapsulating the objects back to processes arise when one is thinking about handling of functions such as adding, multiplying, differentiating or forming sets of functions.

For example, when a student is able to think about the number of cosets a particular subgroup has, can visualize the comparison of two cosets for equality or their cardinalities, or he can operate a binary operation on the set of all cosets of a subgroup, that student is said to understand cosets as objects or has an object conception of cosets (Dubinsky & McDonald, 2001). Also, a student who understands that the standard deviation of a data set is a measure of spread which roughly averages distances from the arithmetic mean would have an object conception of standard deviation (Mathews & Clark, 2007).

2.1.4 Schema

According to Mathews and Clark (2007), "an individual's *schema* is the person's own cognitive framework for a certain piece of mathematics which connects in some way all the ideas that he either consciously or subconsciously views as related to that piece of mathematics" (p. 3). From the point of view of Dubinsky and McDonald (2001), "an individual's schema for a certain mathematical concept is the person's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept" (Dubinsky & McDonald, 2001, p. 3). Schemas are formed when objects and processes that are already constructed are organized in a structured manner, and once they are formed, these objects and processes can be interconnected in various ways. The pieces of schemas already constructed can then be seen as objects and incorporated in the built up of "higher level" schemas and when this happens, we say that the schema has been thematizedto an object (Asiala et al., 1997).

As an example, functions can be formed into sets, operations on these sets can be presented, and properties of the operations checked. All these can then be organized to form a schema for function space which can also be applied to concepts such as dual spaces, spaces of linear mappings, and function algebras (Asiala et al., 1997, p. 8). Also, an individual with a schema for limits can coordinate in some coherent manner his or her cognitive representations of closeness in the domain, understanding of closeness in the range, and conception of function. Mathematical rules, like the chain rule for differentiation, which require the coordination of two or more actions, processes, or objects may also be understood through a schema.

Euclidean Geometry, according to Mathews and Clark (2007), which is also a thematized schema, is an object to a person who knows several geometries, is able to move among them, and can compare and contrast them. Schemas are consequently said to be important to the mathematical empowerment of an individual.

2.2 Statistics Educational Reforms

Over the last two decades, statistics teaching in the American community and the world at large has undergone a number of reforms due to calls from individuals like George Cobb (Aliaga et al., 2005). In America, after the publication of his report, "Heeding the Call for Change: Suggestions for Curricular Action" in 1992 by the Mathematical Association of America (MAA), a lot of inventions were instituted in the teaching of statistics (Aliaga et al., 2005). The GAISE College report by the American Statistical Association (ASA) also offered a list of goals for students in 2002, based on what it meant to be statistically literate (Aliaga et al., 2005). The report presented some recommendations for the teaching of introductory statistics that built on the previous recommendations from Cobb's report (Aliaga et al., 2005).

According to Aliaga et al. (2005), it is anticipated that all introductory statistics courses produce students who are statistically educated. This means students must develop statistical literacy and be able to think statistically. Consequently, they recommended that all introductory statistics courses must:

1. Emphasize statistical literacy and develop statistical thinking.
2. Use real data.
3. Stress conceptual understanding, rather than mere knowledge of procedures.
4. Foster active learning in the classroom.
5. Use technology for developing concepts and analyzing data.

6. Use assessments to improve and evaluate student learning (Aliaga et al., 2005, pp. 14-22).

As a result of these reforms, a remarkable growth has been observed in the number of statistics courses that were taught in colleges and universities in the American community and the world at large, as well as a rise in the number of students who are taking these courses (Aliaga et al., 2005). Those reforms were necessary in this information age, where advancements and innovations in information technology has brought about the collection of huge volumes of data in all sectors of every nation's economy, and knowledge in statistical concepts is required in the analysis of such data.

The goals for students in those reformed courses in the American community, focused on producing students with more conceptual understanding, students who have attained statistical literacy and thinking and those who depended less on learning a set of tools and procedures. The reform aimed at producing students with a thorough grasp of the basic statistical concepts needed to interpret tools and reports from available technology and software in this information age, while minimizing the need to teach the mechanics of the procedures (Aliaga et al., 2005).

2.2.1 The Need for Statistical Literacy, Reasoning and Thinking

Nowadays, the ability to analyze, interpret and communicate information from data has become skills necessary for daily living and to help an individual to be an effective citizen. Concepts in statistics are occupying an increasingly important role in mathematics curricula. Decision-making in society and learning about the world are increasingly being based on evidence derived from a set of data. Statistical methods and ways of thinking are taking over a varied range of human activities such as in psychology, government policy, engineering, health sciences and sustainable environments (Pfannkuch, 2008), and also in the media through numerous kinds of arguments, advertisements, or suggestions (Ben-Zvi & Garfield, 2004a). All these activities are using data to give meaning and understanding to real settings and situations, and statistics education has a crucial role to play in this regard.

Despite the attention given to statistics in school and university curricula, people continue to show signs of poor statistical literacy and reasoning after formally studying statistics as a subject in school (Ben-Zvi 2004; delMas & Liu, 2005; Mathews & Clark, 1997, Armah & Asiedu-Addo, 2014). Rubin (2002) noted that, adults and most students at the college-level do not have an in-depth understanding of data, beyond the simple graphical representations like the bar and pie charts that are often misleadingly presented in the media. It was widely acknowledged by mathematics educators that the foundations of statistical reasoning must be built in the early years of schooling than reserving it for the higher levels like the high school or the university (NCTM, 2000).

According to Rumsey (2002), many statistics instructors agree to the fact that it is the objective of any introductory statistics course is to raise the awareness of students of data in our everyday life experiences and prepare them for a future career in the present information age. To achieve this, statistics educators must work towards achieving two principal goals for our introductory statistics courses. First, students must be trained to be good “statistical citizens,” who understand statistics very well to be able to consume the information that they are flooded with on a daily basis, thinking about it critically, and using it to make good decisions. This, Rumsey stated, some researchers call “statistical literacy.”

The second, and to her, the often underrated goal for our introductory statistics courses, is to develop scientific research skills in our students. This involves the ability to identify questions and problems, collect data, discover and apply tools to interpret it, communicate and exchange results. Though not all our students may conduct scientific studies of their own, it is almost impossible to imagine a student in today's society, not encountering data or statistical results in their course of a career. In every aspect of this scientific method, statistics is involved (Rumsey, 2002).

For an individual to have a deep knowledge of a concept, he is required to be able to recognize counter examples of the said concept. Thus, since it is possible for statistics to be abused in many ways, despite its complexities (Bennett & Briggs, 2002) it is imperative for students who are the future generation to be knowledgeable in statistics so as to analyze data critically and evaluate the information they receive.

There have been strong calls within the past decades for statistics education to emphasize more on statistical literacy, reasoning, and thinking (Aliaga et al. 2005; Cobb, 1992). One such call was from the International Collaboration for Research on Statistical Reasoning, Thinking and Literacy (SRTL). On their homepage, this collaboration points out their goal which is to "foster current and innovative research studies that examine the nature and development of statistical literacy, reasoning, and thinking, and to explore the challenge posed to educators at all levels - to develop these desired learning goals for students". Aliaga et al., (2005) also reiterated the fact that the desired goal of all introductory statistics courses is to produce statistically educated students, which means students must develop statistical literacy and the capability to think statistically. There is the need therefore to distinguish between these terms that are used to refer to the goals of statistics education: statistical literacy, statistical reasoning, and statistical thinking since their definitions and usage among the literature sometimes seem to be coinciding.

2.2.2 Statistical Literacy, Reasoning and Thinking Defined

In a United Nations document, *Making Data Meaningful*, statistical literacy was simply defined as the "ability of an individual or a group to understand and comprehend statistics" (Bencic et al., 2012, p.5). Bencic et al. observed that statistical literacy needs several abilities, most important of which are mathematical and statistical skills, the ability to correctly comprehend the figures and to differentiate between valid and misrepresented data. They further noted that being statistically literate enables people to assess the information that the figures provide and finally to understand what the actual data reveals about society (Bencic et al.).

Aliaga et al., (2005) defined statistical literacy as "understanding the basic language of statistics (e.g., knowing what statistical terms and symbols mean and being able to read statistical graphs) and fundamental ideas of statistics" (p. 14). According to Watson (1997), statistical literacy is "the ability to understand and critically evaluate statistical results that permeate our daily lives - coupled with the ability to appreciate the contributions that statistical thinking can make in public and private, professional and personal decisions" (p. 14).

From these definitions, it is evident that for an individual to be statistically literate, he must be able to comprehend and interpret statistical concepts, be able to evaluate critically and extract information from statistical procedures and presentations in everyday life and be able to identify and question assertions made without statistical base. This was what Watson (1997) observed when he described statistical literacy along a three-tiered continuum as: "a basic understanding of probabilistic and statistical terminology; an understanding of probabilistic statistical language and concepts when they are embedded in the context of wider social discussion; and a questioning attitude which can apply more sophisticated concepts to contradict claims made without proper statistical foundation" (p. 2). Statistical literacy is an important skill expected of all citizens in an information-dense societies, and it is often seen as an expected outcome of schooling (Gal, 2002).

Rumsey (2002) observed the phrase "statistical literacy" to be too broad. So in an attempt to clarify issues she omitting the phrase "statistical literacy" from her discussions and instead used two distinct phrases to refer to the two distinct learning outcomes expected of every introductory statistics course: . "Statistical competence" which denotes the basic knowledge that underlies statistical reasoning and thinking, and "statistical citizenship" which is the ultimate goal of helping an educated person to develop his abilities in order to function effectively in today's information age.

For students to achieve both of these important goals, Rumsey (2002) believed it was important for them to understand and use statistical ideas at many different levels. To start with, there is the need for a certain level of competence, or understanding, of the fundamental ideas, terms, and language of statistics. Not only this, but rather to become good statistical persons or citizens and research scientists, one must have the ability to explain, make judgments, evaluate, and take decisions about the information he receives. There is therefore the need for additional skills in statistical reasoning and thinking, whose foundation is first developed at the statistical literacy level.

Ben-Zvi and Garfield (2004a) also defined statistical reasoning as "the way people reason with statistical ideas and make sense of statistical information" (p. 7). It encompasses the ability to make interpretations on the basis of sets of data, graphical representations, and statistical summaries. With a store of knowledge in statistical reasoning, one can combine ideas about data and chance, and be able to give explanations of statistical processes and interpret statistical results fully (Ben-Zvi & Garfield, 2004a).

Statistical thinking also entails an understanding of the why and how of statistical investigations as well as the "big ideas" behind statistical investigations. These consist of the omnipresence of variation and knowing the right tools to use for data analysis. According to Gal, and Garfield (1997), it comprises recognizing and understanding the entire statistical investigative process (from posing a question to collection of data to selecting the type of analyses to testing assumptions, etc.), understanding how to use models to simulate random phenomena, understanding how data are obtained to estimate probabilities, being acquainted with the how, when, and why inferential tools can be used, and the ability to understand and utilize the context of a problem to plan and evaluate investigations and to come out with conclusions.

From these definitions it could be seen that statistical literacy is essential for statistical reasoning and thinking. An individual must have a conceptual understanding of basic statistical concepts before being able to use them to interpret and evaluate statistical representations and summaries. She will then be in the best position to collect an appropriate data to solve problems or judge a statistical analysis.

2.3 Teaching for Understanding

In the teaching and learning of mathematics it is important to ensure students' understanding of the objects of mathematics presented to them. It is the expectation of all teachers that students retain what they are taught after completing their courses and writing their examinations.

It is an undeniable fact that education must be about teaching for understanding, but we can hardly say that this goal is being achieved in our schools. While the traditional lecture method of teaching, exercises, and drills can help students memorize facts and formulas and get the right answers on tests, this age-old style of teaching does not help students attain an in-depth understanding they need of complex ideas and apply the knowledge gained in new settings or situations. Brandt (1993) observed that many students, even at the college level, find it difficult to understand, numerous basic concepts taught in school. That is, they are unable to use knowledge learned in a setting and appropriately apply in a different setting" (Brandt, 1993).

According to Kickbusch (1996), critics of public education had argued since 1983 that quite a number of American students lack the kind of knowledge or skills to warrant either personal life success or national economic competitiveness. What has caused these critics to be alarmed had been the apparent inability of many to be involved in complex problem solving activities as well as in applying knowledge and skills gained in school to solve real life problems in workplaces. Since the traditional measures of school outcomes and standardized achievements do not require students to apply knowledge in new settings, they said it should not be surprising that students fail to meet such expectations.

McTighe and Seif (2011) opined that a synthesis of cognitive research supports the notion that when one has a deep understanding of a subject matter, the individual can transform factual information into usable knowledge. On the other hand, when knowledge is learned by rote, it is rarely transferred. Rather, transfer occurs mostly when the learner knows and understands the underlying concepts and principles that can be applied to problems in new contexts. McTighe and Seif (2011) therefore noted that if learners learn with understanding, it is more likely to promote transfer and application than just memorizing information from a text or a lecture.

Hiebert and Lefevre (1986) described understanding to be the state of knowledge when new mathematical information is connected appropriately to an existing knowledge. In a related work, Wiggins and McTighe (2005) described understanding and proposed that understanding is shown through six facetsthat offer different types of evidence of understanding. They stated that when we really understand, we:

- can explain—via generalizations or principles, providing justified and systematic accounts of phenomena, facts, and data; make insightful connections and provide illuminating examples or illustrations.
- can interpret—tell meaningful stories; offer apt translations; provide a revealing historical or personal dimension to ideas and events; make the object of understanding personal or accessible through images, anecdotes, analogies, and models.
- can apply—effectively use and adapt what we know in diverse and real contexts—we can "do" the subject.
- have perspective—see and hear points of view through critical eyes and ears; see the big picture.
- can empathize—find value in what others might find odd, alien, or implausible; perceive sensitively on the basis of prior direct experience.
- have self-knowledge—show metacognitive awareness; perceive the personal style, prejudices, projections, and habits

> of mind that both shape andimpede our own understanding; are aware of what we do not understand; reflect on the meaning of learning and experience (p. 84).

McTighe and Seif (2011) opine that an essential goal of schooling is to teach for meaning and understanding. According to them teaching for meaning and understanding are two sides of the same coin and they both occur when students are taught to be able to explain and interpret ideas, puts facts into a larger context, investigate into essential questions and apply their learning in authentic situations. Included in the six principles of the US based NCTM's Principles and Standards for School Mathematics is that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p. 11).

The NCTM (2000) stated that in learning mathematics, if factual knowledge and procedural proficiency are aligned with conceptual knowledge, it helps students to become effective learners. Students will learn to know the significance of reflecting on how they think and thus, learn from their mistakes. It will also help students to become capable and assertive in the way they tackle difficult problems and be ready to persevere when they face challenging tasks.

Broers (2009) differentiated between conceptual knowledge and propositional knowledge. The propositional knowledge he described as "isolated knowledge of definitions, principles and basic ideas" (p. 2). Further, he pointed out that as soon as knowledge becomes interrelated or structured, propositional knowledge becomes conceptual understanding. This is so because these interrelationships encompass organizing mentally, the links between concepts. Apart from showing relationships between concepts, a knowledge network that shows conceptual understanding also demonstrates a hierarchical structure, with higher level concepts consisting of more elementary ones.

Hiebert and Lefevre, (1986) defined conceptual knowledge as knowledge that "… is rich in relationships" (p. 3). "… Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network" (p. 4). They also noted that "… a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information" (p. 4).

From this definition, Groth and Bergner (2006) saw the essence of conceptual knowledge as that which involves "forming cognitive connections between bits of information that might otherwise be perceived as unrelated" (p. 39). Balka, Hull, and Miles (n.d.) observed a major similarity in the definition of conceptual understanding put forth by both the National Council of Teachers of Mathematics (NCTM) and the National Research Council (NRC):

> Students demonstrate conceptual understandingin mathematics when they provide evidence that they can recognize, label, and generate examples of concepts; use and interrelate models, diagrams, manipulatives, and varied representations of concepts; identify and apply principles; know and apply facts and definitions; compare, contrast, and integrate related concepts and principles; recognize, interpret, and apply the signs, symbols, and terms used to represent concepts. Conceptual understanding reflects a student's ability to reason in settings involving the careful application of concept definitions, relations, or representations of either. (p. 2)

There have been numerous concerns raised in Ghana by all stakeholders of education on the poor performance of students in mathematics. Prof Sitsofe Anku, the

Director of the Meagasa Mathematics Academy, organized a national training of trainers of mathematics in Accra for selected basic and JHS teachers under the theme "Revamping Mathematics Education in Ghana through Transformation". The training which took place from the 27th of April to 8th May, 2015 was necessitated because there had been a massive failure in mathematics from basic to the tertiary level and that had turned out to be a national worry which needed a holistic approach to address the menace (Ghana News Agency [GNA], 2015).

Professor Anku, in his comments during the training said teachers are to be partly blamed for students' failure in mathematics. This is because a lot of mathematics teachers failed the subject during examinations while in school and have decided to use teaching as the last option to earn money. Since the teacher himself has no interest in the subject, it affects his way of teaching and in the long run the students he is teaching. He said if teachers are knowledgeable in mathematics, it will not be difficult for students to understand and appreciate the subject and in so doing make a positively impact on their performance. He added that it is necessary for teachers to show understanding of the principles and methods of mathematics, in order to teach students for easy apprehension (GNA, 2015).

He further commented on the country's curriculum which he observed was only geared towards getting students ready for examinations and that their outfits only prepared students to be analytical and tackle real life problem solutions. He therefore encouraged students to appreciate the study of mathematics, since mathematics is used in every profession (GNA, 2015).

2.4 Mathematical Knowledge

Mathematical knowledge has generally been put under two main groups and different researchers have used different terms to characterize them at different times and in different situations with some overlapping and others showing slight differences. Skemp (1976; 1989) used relational understanding and instrumental understanding. According to him, relational understanding encompasses knowing what and why something happens the way it does. It is not just knowing an action or rule, but also knowing the justification for it. Conceptual structures are constructed in a way that produces unlimited plans to help a child make alternatives at every stage in relational learning. On the other hand, instrumental understanding involves using rules or procedures without understanding how they work (Skemp, 1989). With this type of understanding in mathematics, children see mathematics as a bunch of rules which must be applied when necessary.

Byers and Herscovics (1980) also differentiated between intuitive and formal understanding of mathematics. With intuitive understanding, they meant the type that occurs when a student is able to solve a problem without first analyzing the problem. In such an instance a student solves a problem correctly using a wrong means. On the other hand, when a student is able to connect mathematical symbols and notations with the relevant mathematical ideas and combine them into chains of logical reasoning he is said to possess formal understanding.

Hiebert and Lefevre (1986) used conceptual knowledge and procedural knowledge to describe this duality of mathematical knowledge. As stated earlier, he defined, conceptual knowledge as one characterized very clearly as knowledge that is rich in relationships. Hiebert and Lefevre, (1986) opined that one way conceptual knowledge can be developed is by the construction of relationships between pieces of information. This can either be between two pieces of information already stored in the memory or between an already stored knowledge and one newly learned. Another way of developing conceptual knowledge is by creating relationships between knowledge that already exist and new information received into the system. This, they likened to Piaget's process of assimilation where new information is blended with an already existing knowledge structure or networks. In this, the new information becomes part of the already existing network (Hiebert & Lefevre, 1986).

According to Hiebert and Lefevre (1986), procedural knowledge consists of two main parts. The first part is made up of the formal language, or symbolic representation system of mathematics. This involves becoming familiar with the symbols used to represent mathematical ideas and becoming aware of the syntactic rules for writing rules in an acceptable form. They noted that this does not denote knowledge of meaning, but rather an awareness of surface features. The second part consists of the algorithms, rules, or procedures for completing mathematical tasks. That is a step by step instructions on how mathematical tasks are completed. What the authors emphasized is that both conceptual and procedural knowledge must be linked, else "students may have a good intuitive feel for mathematics, but not solve the problems, or they may generate answers but not understand what they are doing," (p. 9).

Whiles procedural knowledge is associated with the ability to perform procedures, conceptual knowledge is associated with the knowledge of relationships. Procedural knowledge depends much on computational skills and application of procedures within different representation forms, it does not necessarily need an in-depth understanding of the underlying concept. Procedural knowledge often relies on automated procedures and unconscious steps, while conceptual knowledge requires conscious thinking (Hiebert & Lefevre, 1986).

Balka, Hull and Miles (n.d.) observed that for decades, prominence has been given to procedural knowledge in school mathematics or what is being referred to currently as procedural fluency. Rote learning has been the norm, with little attention given to the understanding of mathematical concepts. The authors continued that currently, much effort have been put in place to concentrate on students' understanding of the mathematics they learn or what can be termed as mathematical proficiency. In their document "Adding It Up: Helping Children Learn Mathematics", Kilpatrick, Swafford and Findell (2001) listed five strands of which conceptual understanding is included. The strands they noted, are intertwined and interdependent and they include the notions suggested by the NCTM in its Learning Principles. To be mathematically proficient, or learn mathematics successfully, Kilpatrick, Swafford and Findell (2001) noted that an individual must have:

- Conceptual understanding: comprehension of mathematical concepts, operations, and relations,
- Procedural fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately,
- Strategic competence: ability to formulate, represent, and solve mathematical problems,
- Adaptive reasoning: capacity for logical thought, reflection, explanation, and justification and
- Productive disposition: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 5).

Kilpatrick, Swafford and Findell (2001) indicated that all the five strands listed above are crucial if students are to understand and use mathematics. The authors, defined conceptual understanding as “an integrated and functional grasp of mathematical ideas” (p. 118).

Balka, Hull and Miles (n.d.) gave some examples of activities a child will do to demonstrate conceptual understanding. These included the following:

- In grades 5 through 6, operations with decimals are common topics. So for a child solving the problem: "What is 6.345 x 5.28?", he/she will demonstrate conceptual understanding when he or she can explain that 335.016 cannot possibly be the correct product since one factor is greater than 6 and less than 7, while the second factor is greater than 5 and less than 6; therefore, the product must be between 30 and 42.
- For grades 1 through 4, basic facts for all four operations are major parts of the mathematics curriculum. So for a child to solve: "What is 6 + 7?" Even though computational fluency is what is expected of the student, an initial explanation might be "I know that 6 + 6 = 12; since 7 is 1 more than 6, then 6 + 7 must be 1 more than 12, or 13." Similarly, for multiplication, "What is 6 x 9?" "I know that 6 x 8 = 48. Therefore, the product 6 x 9 must be 6 more than 48, or 54." (p. 2).

Balka, Hull and Miles (n.d.) also showed how teachers could help their students to be mathematically proficient. They noted that:

- Getting students to use manipulatives to model concepts, and then verbalize their results, assists them in understanding abstract ideas.
- Getting students to show different representations of the same mathematical situation is important for this understanding to take place.
- Getting students to use prior knowledge to generate new knowledge and to use the new knowledge to solve problems in unfamiliar situations is also crucial for conceptual understanding.
- Getting students to see connections between the mathematics they are learning and what they already know also aids them in conceptual understanding (p.4).

As noted by Kilpatrick, Swafford and Findell (2001), when students have a conceptual understanding of the mathematics they have learnt, it helps them to "avoid many critical errors in solving problems, particularly errors of magnitude" (p. 120). Balka, Hull and Miles (n.d.) concluded by quoting the NCTM Principles: “Learning with understanding is essential to enable students to solve the new kinds of problems they will inevitably face in the future” (p. 4).

Presently, in a mathematics classroom concepts are taught first and foremost. Even though procedures are also learnt they are not done without a conceptual understanding. One main importance of stressing conceptual understanding is that it is less likely to forget concepts than procedures. As a result, if a person has conceptual understanding, he can always reconstruct the process that may have been forgotten. On the other hand, if the person has only a procedural knowledge, reconstruction of the forgotten procedure becomes impossible. In mathematics, conceptual understanding with procedural skill, is much powerful than only procedural skill (Schwartz, 2010).

2.5 School Mathematics and its Teaching in Ghana

Prior to Ghana’s independence from British rule in 1957, arithmetic, which was mainly mechanical number facts and tables of measurements was the mathematics studied in Ghanaian elementary schools. As a result, the textbook in use by then, ‘Larcombe’s Arithmetic series’was purposely designed to promote and develop good mental skills in all students. The textbook was characterized by the speed test in mental arithmetic. However, at the secondary level, arithmetic, algebra and geometry were taught (Mereku, 2010).

As a result of the discovery of new mathematics in the 1960's, the African continent, and as such Ghana, underwent curricular changes in mathematics education. After World War II, it became necessary for the school system to be shifted from an 'elitist' to a 'comprehensive' one which led to the expansion of school facilities and student populations. As such, it became necessary for the school mathematics curriculum to be re-organized to also make it more comprehensive to satisfy the needs of the growing number of students in the schools (Mereku, 2010).

According to Mereku (2010), new systems and policies of education were searched for to help develop rapidly the human resources of the new nations in Africa. Hence, after many conferences and workshops, many projects were launched which introduced new textbooks into the nations. Some of which included the Entebbe Modern Mathematics, the 'New Mathematics for Primary Schools (NMPS)', which was to make the learning of mathematics more interesting and more meaningful to Ghanaian children, the Joint Schools Project (JSP), which aimed to produce new mathematics course for West African secondary schools up to school certificate level.

The movement for changes in the content of school mathematics led to the new name 'modern mathematics' which was to permit an approach to mathematics which will facilitate the learning of basic language and structure of mathematics quickly. It was the main aim of the 'new mathematics' to ensure a connection between school mathematics and university mathematics. As a result, there were major changes in pre-university mathematics education in Ghana in the late 1960's and the early 1970's (Mereku 1999).

2.5.1 School Mathematics and the 1987 Reforms

There was therefore an educational reform in 1987 in Ghana which saw the shortening of the duration for pre-university education from 17 to 12 years. This led to the development of new syllabi for mathematics for primary, junior and senior secondary mathematics in 1988 and a total re-organization of their contents to a teaching syllabus (Mereku, 2010). However, Mereku (1999) observed that in the 1988 JSS syllabus, attention was not given to the change in the educational system, since at the primary and junior secondary level, the content of the syllabi were just designed to cup tie with the content of the existing textbook, the Ghana Mathematics Series (GMS) schemes which had already been developed a decade earlier. Also, compared to the content and difficulty level of the school mathematics before the reform, that of the reform were raised far above it and was heavily loaded. Furthermore, the syllabi at the secondary school (both JSS and SSS) were designed to be used by all students, irrespective of whether they will continue doing mathematics at the higher level or they will just need it to understand their environment, to function in it and help in its development (Mereku, 1999). The later, Mereku (2000) observed as the main factor leading to the poor performance of students in mathematics the decade after the 1987 educational reforms.

The teaching and learning of mathematics, especially at the basic level, after the 1987 reform was not so different from what existed in the 1970s and the 1980s, as the traditional method, that gave emphasis to computational skills was primarily in use. Though new topics were introduced and were reflected in the textbooks, the GMS schemes, teachers were not adequately prepared to handle the new methodologies which had been developed by psychologists to be used in teaching them. As a result, many JSS teachers faced challenges in teaching some of the topics in the modern mathematics in the syllabus (Mereku, 2010). Mereku (1999) cited a study ordered by the Ministry of Education within the first five years into the reforms which also revealed the low attainment of students in mathematics in the public schools despite the trainings given to teachers. From the study, there was also an indication that the teaching of mathematics in the basic schools was characterized by emphasizing computation skills, the learning of formulas, rote learning and teaching as telling. As a result of the type of mathematics education students went through at the basic level, they entered the secondary schools having a very weak basic foundation in school mathematics.

At the secondary school level, Doku, cited by Mereku (2010) also observed the lowering of the quality of mathematics education as a result of the overloading of the content of the syllabus. Since there was also much focus on preparing students for their final examination, the Senior Secondary School Certificate Examinations, than attaining mathematical skills, students were observed to be deficient in basic mathematical skills. Also, as a result of the competitive teaching and learning of mathematics in the schools, Doku, cited by Mereku (2010) reported that teachers concentrated on the brighter students than in assisting the low ability ones.

2.5.2 The Culture of Mathematics Teaching and Learning

Though there has further been revisions in the mathematics syllabus in the years 2001, 2007 and 2012 for Ghanaian basic schools and 2003, 2007 and 2012 for the secondary schools, all with the aim of improving on the teaching and learning of mathematics, not much has changed in the way teachers teach the subject. Fletcher (2005) argued that though mathematics educators in Ghana wished to follow a constructivist approach in the teaching and learning of mathematics, teachers at the basic and senior secondary schools continued to give too much prominence to memorization and 'imitation' instead of helping students to understand and explain.

Based on their amassed observations and experiences of mathematics education in Ghana and outside Ghana, Fredua-Kwarteng and Ahia (2004), found four major causes of the problem of mathematics learning. Among the four were “the culture of mathematics teaching” and “the culture of mathematics learning”. According to them, a key contributor to students’ dislike of mathematics is the culture of mathematics teaching. Fredua-Kwarteng and Ahia (2004) share the belief that the main purpose of teaching is to cause students to learn. As such, if Ghanaian students are not learning mathematics as they should, then it can be concluded that their teachers have not as yet found effective ways of teaching mathematics to them. They observed that teachers who are weak in mathematics produce weak mathematics teachers.

Based on their experience as people who have both taught mathematics in Ghanaian schools and observations made in Ghanaian schools, as well as contacts with some teachers of mathematics in Ghana, Fredua-Kwarteng and Ahia (2004) identified the following as some of the culture of mathematics teaching in Ghana:

1. Mathematics teachers normally appear before their classes, give a definition of a mathematical concept, work a few examples from the mathematics text on the chalkboard and at the end of the instruction assign students some exercises to do. In other words, mathematics teachers act before a passive audience that is supposed to absorb the knowledge transmitted.
2. Usually when students do not understand a teacher’s method of presenting a mathematical concept, the teacher would not change the method of presentation. Instead, the teacher would blame the students for being lazy or unintelligent. Thus, the students are required to learn the teachers’ method whether they like it or not.

3. Mathematics teachers are mostly interested in answers or solutions to mathematical questions or problems rather than the processes or methods used to obtain the answers or solutions. Teachers simply give answers or solutions to their students without first having the students worked for those solutions.

4. Mathematical concepts are taught as objective, discrete facts without linking them together. For example, elementary school teachers do not demonstrate to their students that there is an affinity between decimal, ratio, rate, fractions, percentage, and proportion.

5. Students are hardly encouraged to ask questions, make comments or suggestions about what is being taught. Students' primary responsibility is simply to listen passively to the teacher, take notes when necessary, and store the knowledge in the dustbins of their brains, so to speak. The teacher brands students who dare to ask questions or disagree to a particular solution of mathematical question "challengers" or a threat to his/her authority in the classroom. Such students would suffer harassment from the teacher and possibly isolation from their peers who might label them "too known". This atmosphere of fear and hostility is not conducive to effective mathematics learning. And it may be one of the greatest sources of epistemological anxiety in mathematics learning and a major cause of mathematics underachievement in Ghanaian schools.

6. Mathematics teaching is decontextualized. Teachers hardly connect mathematical concepts that they are teaching to the lives of their students or cultural practices in our society. Mathematics is taught as if it has no social or economic referents or relevance in our society. For instance, a teacher is teaching her students how to calculate simple interest, but fails to talk about traditional arguments for and against charging interest and the fact that interest is normally quoted in a non-percentage term.

7. Mathematics teachers simply give formulas or algorithms to their students to use in doing mathematics. Normally, the underlying logic or philosophy of the formulas or algorithms is not explained to the students. In fact, students are required to regurgitate the formulas or algorithms and recall them during tests, quizzes, class assignments, home assignments, and examinations. Mathematics learning becomes a matter of memorization.

8. Mathematics is taught without using any other materials, except chalk and chalkboard. If mathematics is the study of patterns and quantitative relationships--- arithmetic and number theory study the pattern of numbers and counting; geometry studies the patterns of shape; calculus allows us to handle patterns of motion; logic studies the patterns of reasoning; probability deals with patterns of chance--- one would have expected that teachers use other materials to enhance the teaching of mathematics. Admittedly, at the primary 1,2 and 3 level teachers encourage students to use stick counters, marbles and their fingers or toes to do the four basic operations of mathematics.

10. Mathematics teachers have a hidden assumption that only the most brilliant students are capable of learning mathematics. So that students who experience difficulties understanding mathematical concepts are left behind to fend for themselves, while the so-called brilliant few are motivated by provision of extra assistance.

11. English language is primarily the exclusive means of instruction, even if the teacher speaks the same indigenous language as the students or majority of the students. The students might not have attained any proficiency in English language in order to process mathematical ideas efficiently. The teacher may also have difficulties expressing mathematical ideas precisely in English.

13. Students in elementary schools are usually drilled mentally on the multiplication table, addition and subtraction facts under the label "mental". Some students are made to recite the times table in a parrot-like fashion in the belief that once mastered it would facilitate the learning of other mathematical concepts.

According to Fredua-Kwarteng and Ahia (2004), the way Ghanaian students learn mathematics, which they referred to as "the culture of mathematics learning" reflects the way mathematics teachers teach the subject (the culture of mathematics teaching) and the perception students have about the nature of mathematics and its importance in formal education. They observed the following generalization to be the culture of mathematics learning in Ghana:

1. Students learn mathematics by listening to their teacher and copying from the chalkboard rather than asking questions for clarifications and justification, discussing, and negotiating meanings and conjectures. Consequently, students learn mathematics as a body of objective facts rather than a product of human invention.
2. Students hardly read their mathematics textbooks or other mathematics textbooks. Where students read the prescribed mathematics textbooks, they read them like the way they read novels or newspapers.
3. Students could go to the library to read newspapers or novels, not mathematics. Mathematics is learned only in the mathematics classrooms or for examinations, quizzes, or tests.
4. Students could form a small study group outside of their classroom to do homework, assignments or prepare for an examination or tests, but not for discussing mathematical concepts that were taught to them in the classrooms.
5. Students learn mathematics by regurgitating facts, theorems or formulas instead of probing for meaning and understanding of mathematical concepts. That is to say, students hardly ask the logic or philosophy underlying those mathematical principles, facts, or formulas.
6. Students accept whatever the teacher teaches them. The teacher is the sole authority of mathematical knowledge in the classroom, while the students are mere receptors of mathematical facts, principles, formulas, and theorems. Thus, if the teacher makes any mistakes the students would also make the same mistakes as the teacher made.
7. Most students do mathematics assignments and exercises not as a way of learning mathematics, but as a way of disposing off those assignments to please the teacher. This implies that mathematics assignments are not construed as an instrument for learning mathematics.
8. Students go to mathematics classes with the object to calculate something. Therefore, if the classes do not involve calculations they do not think that they are learning mathematics. So students learn mathematics with the goal to attain computational fluency, not conceptual understanding or meaning. For a conceptual understanding requires students to think critically and act flexibly with what they know. Students are fond of asking, “How do you

calculate that?" instead of asking "why do you calculate it in that way?".

9. Students learn mathematics with the aim to pass a test or examination. After passing the test or examination, mathematics is no longer of importance to the students.
10. Students have internalized the false belief that mathematics learning requires an innate ability or the "brains of an elephant".
11. It is generally believed that only science-oriented students must learn and master mathematical principles, not so-called arts or business students. Alternatively, most people (including some mathematics teachers) believe that art or business students require a pass in mathematics in their final examinations. Though people believe that artisans or technicians must learn mathematics, they don't believe that they have to master as much mathematics as science students (those who want to study engineering, medicine, architecture, computering, electronics, etc.).

In Ghana, students' low attainment in mathematics has been an issue of concern for many years now (Mereku, 2003). This has been seen in their performance in the report on the Trends in International Mathematics and Science Study (TIMSS) since 2003. TIMSS is a series of assessments conducted by the International Association for the Evaluation of Educational Achievement (IEA) once every four years to assess students in mathematics and science internationally. Four main content areas are assessed in mathematics. These are: Number, Algebra, Geometry, and Data and Chance, and they are assessed under three main cognitive domains: Knowing, Applying and Reasoning. Since Ghana's participation for the first time in 2003, the overall performance of Ghanaian junior high school two (JHS2) students has consistently been very poor. In 2003, their performance placed them in the 45th position out of forty six participating countries on the overall mathematics achievement results table with only 2% and 9% reaching the intermediate and low international benchmarks respectively (Anamuah-Mensah, & Mereku, 2005).

In 2007, though the performance of Ghana's JHS2 students' improved significantly (from a scale score of 276 in 2003 to 309 in 2007), it remained among the lowest in Africa and the world (Mullis, Martin, & Foy, 2008). Their performance on the international benchmarks also improved significantly with 4% and 17% reaching the intermediate and the low international benchmarks. Their mean score of 309 placed Ghana at the 47th position out of 48 participating countries on the overall mathematics achievement table (Mullis, Martin, & Foy, 2008).

Similarly, in 2011, there was an improvement over the performance in 2007. However, their average score of 331 placed Ghana last on the list of 59 participating countries - including the benchmarking participants–(Mullis, Martin, Foy, & Arora, 2012). It was also reported that the percentage of Ghanaian JHS2 students' with achievement too low for estimation exceeded 25%. With regards to their performance on the international benchmarks, there was also an improvement. Twenty one percent (21%), 5% and 1% of the students reached the low, intermediate and the high international benchmarks respectively. Interestingly, it was noted that among the four content areas assessed, the lowest mean score for Ghanaian students was recorded in Data and Chance (Statistics) (Mullis, Martin, Foy, & Arora, 2012).

The Organization for Economic Co–operation and Development (OECD) in their document on 'Measuring Students Knowledge and Skills' wrote that their mathematical literacy domain is concerned with the ability of students to rely on their mathematical competencies to meet the challenges of the future. This they said is concerned with students' abilities to "analyze, reason, and communicate ideas effectively by posing, formulating and solving mathematical problems in a variety of domains and situations" (OECD, 1999, p. 41). The organization also defined mathematical literacy as:

> An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgments and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen. (OECD, 1999, p. 14).

2.6 Studies in Students' Conception of the Arithmetic Mean

One of the statistical summary measures that students first come across in their study of mathematics is the arithmetic mean or simply the mean. This is because the procedure for finding the arithmetic mean of a data set is so simple, requiring only two computational skills: addition and division. As a result, students are introduced to it around the primary four or five of the educational ladder. However, at that level, the concept is not presented to students as a statistical measure, but rather as an application of division (Bremigan, 2003). Thus, in the classroom situation, students and teachers alike might be tempted to think that when an individual is able to compute the arithmetic mean of a set of data successfully, then that individual is said to have a conceptual understanding of it. There is, however the possibility of an individual to calculate the arithmetic mean without actually knowing what it represents, its relation to the other values in the data set as well as its relation to other measures of center or spread of the data set (Zawojewski, & Shaughnessy, 2000).

Mathews and Clark (2007) conducted a study whose aim was to determine the conceptions of the arithmetic mean, standard deviation and the Central Limit Theorem that most successful students had shortly after completing a statistics course. They observed that even though all the eight students they used as participants could correctly calculate the arithmetic mean of a data set, many of them confused it with the mode as well as the population and sample proportions when they had to discuss specific examples. They also found that not a single participant had a mature understanding of standard deviation.

In their study to analyze students' and teachers' conceptions and misconceptions of average (arithmetic mean), Magina, Cazorla, Gitirana, and Guimarães (2008) also witnessed that teachers and students in general, found the average (arithmetic mean) as a difficult concept to understand. They observed a presentation of conceptions with no statistical validity. For example, they found a larger number of primary four pupils and a reduced number of undergraduate students confusing the average (arithmetic mean) with the sum of the values of a data set. Another common misconception they noticed was students confusing the average (arithmetic mean) with the maximum value of a data. This was related to the lack of comprehension of the property that the average (arithmetic mean) can only take on values between the extremes (Strauss & Bichler, 1988).

Uccellini (1996) cited a case that when a group of middle school students is asked to find the arithmetic mean of a set of numbers, they probably will give the correct answer. However, when the same students are asked the relationship between their answer and the set of numbers the response usually given is that of the algorithm "add them up and divide". This observation was also witnessed in a study conducted by Armah and Asiedu–Addo (2014) among first year undergraduate students prior to their introductory statistics course.

The issue of the arithmetic mean arises whenever there is the need to summarize or describe a set of data. Most research works report on the arithmetic mean and inferential statistics deals with means and difference between means most of the time. The arithmetic mean is not only an important concept in statistics but also an important concept for informed citizens (National Council of Teachers of Mathematics [NCTM], 1989). Its significant role among averages is indisputable and it helps in shaping a statistically literate society. Mokros and Russell (1995) identified it as crucial to statistical understanding as well as mathematical significance.

Due to the cross-disciplinary characteristic of the arithmetic mean, it makes it conceptually constructive in diverse disciplines of study, including statistics, mathematics, and physics, and the way it is used as a tool for statistics it ever-present in educational, vocational, and recreational settings. The arithmetic mean's diversity has promoted research targeted at finding an understanding of how students arrive at their knowledge base for it and the instructional techniques that promote its conceptual learning (Marnich, 2008).

Though the concept of the arithmetic mean seems so simple, previous studies have shown that students do not have a deeper understanding of it as well as the properties relating it (Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988; Mathews & Clark, 2007; Chatzivasileiou, Michalis, & Tsaliki, 2010). Students have been found to face difficulties when using the "add them up and divide" algorithm to solve problems involving the weighted mean, which entails more than a direct application of this algorithm (Pollatsek, Lima, & Well, 1981; Callingham, 1997) such as not being able to take frequencies into account when solving weighted average problems. The misconceptions students displayed in these studies according to Cai (1998) were not as a result of their lack of procedural knowledge in the computation of the arithmetic mean, but rather, their lack of conceptual understanding of the concept, thereby affecting their ability to use the algorithm.

Since the arithmetic mean is both an important concept in statistics and a computational algorithm, its conceptual understanding must include both an understanding of the computational algorithm as well as its statistical nature. With regards to the statistical understanding of the arithmetic mean, students must show the ability to use it to summarize and make meaning of data sets or to compare data sets. With its conceptual understanding, students should be able to apply it to solve problems correctly. In other words, students should not only demonstrate how to compute the arithmetic mean using the algorithm but also know when and how to correctly apply the procedure to problem situations (Cai, 1998).

Students are known to intuitively construct a sense of mode and median before being formally introduced to the concepts and procedures for finding them. On the other hand, they seldom have the opportunity or insight to do the same with regards to the arithmetic mean (Marnich, 2008). Strauss and Bichler (1988) analyzed the concept of the arithmetic mean into its properties and explored the development of children's understanding of these properties. They arrived at seven of these properties that students must comprehend, and indicated that the concept of the arithmetic mean is closely related to the comprehension of these properties. The properties, according to Strauss and Bichler (1988) are:

A. The arithmetic mean is located between the extreme values,
B. The sum of the deviations from the arithmetic mean is zero,
C. The arithmetic mean is influenced by values other than itself,
D. The arithmetic mean does not necessarily equal one of the values that was summed,
E. The arithmetic mean can be a number which does not have a counterpart in the physical reality,
F. When one calculates the arithmetic mean, a value of zero, if it appears, must be taken into account.

G. The arithmetic mean is a representative value of the data from which it has been calculated (p. 66).

Leon and Zawojewski (1990) conducted an investigation into four of these properties using 4th grade, 8th grade and college students. The four properties used were properties A, B, F and G above. Their results indicated that students found out that the difficulty level for properties F and G are nearly two times as compared with that of properties A and B, which Strauss and Bichler (1988) termed ‘the statistical properties’. In trying to explain this observation, Leon and Zawojewski (1990) stated among other reasons that, the first two properties deal with the idea of data distribution and its relationship to the arithmetic mean which students find easy to understand, whereas the last two addresses the representative and abstract interpretation of the arithmetic mean. Secondly, they noted that the first two properties were computationally based hence students could solve it by applying knowledge about number and operations whereas the last two do not. This observation supports other studies that have equally argued that most students find it comparatively easy to understand the arithmetic mean as a computational construct, but relatively difficult to understand it as a representative value (Pollatsek et al., 1981; Mevarech, 1983; Goodchild, 1988; Chatzivasileiou, Michalis & Tsaliki, 2010).

Mokros and Russell (1995) also distinguished between two basic models used in the literature when defining the arithmetic mean: *fair share* and *balance*. The authors indicated that research works about these models focused on the arithmetic mean, and most importantly on just a way of making sense of it, but overlooked the idea of representativeness as an important characteristic of the arithmetic mean. Mokros and Russell (1995) saw the arithmetic mean as indicating the center of a distribution and this is useful in summarizing, describing, and comparing different data sets. According to them, a powerful conception of the arithmetic mean is that of a "mathematical point of balance." From this viewpoint, the different contributions of a set of data set are balanced with one another, and their equilibrium point becomes a distinctive characteristic of the collection. Mokros and Russell (1995) however, are of the view that, a well-constructed idea of representativeness must comprise an understanding of the arithmetic mean and how it works.

A previous study by the same authors (Russell & Mokros, 1990) indicated that younger children could construct the idea of the average to mean a reasonable indicator of center, which is an important base block in learning about mathematical representativeness. They stated that the idea of representativeness was not something to superimpose children's calculation skills and they disagreed with researchers (Strauss & Bichler, 1989; Leon & Zawojewski, 1991) who claimed that representativeness was a concept children could only develop after they have grasped the statistical aspects of the arithmetic mean. Rather, in actual sense representativeness is an idea that could be constructed early. This view was contradicted in a study (Mokros & Russell, 1995) in which the same authors concluded by stating that their work with children and adults as well has led to the realization that the "arithmetic mean is a mathematical object of unappreciated complexity (belied by the "simple" algorithm for finding it) and that it should only be introduced relatively late in the middle grades, well after students have developed a strong foundation of the idea of representativeness" (Mokros & Russell, 1995. p. 38).

Learning about the arithmetic mean, Mokros and Russell indicated, is one of the first encounters students have with a mathematical construction which expresses a connection between some numbers, and this connection among a data set, reflected in the arithmetic mean, is an abstract mathematical construction with no specific referent in the real world. Thus summarizing a data set by finding an arithmetic mean demands a manipulation that accounts for, but at the same time submerges, the concrete data points (Mokros & Russell, 1995).

They found that most students knew the 'add them up and divide' algorithm but are not able to use it in any meaningful way, as observed by Cai and Moyer, (1995). Most of the Grade 6 students Cai (2000) studied had difficulties reversing the algorithm for the arithmetic mean. They rather found it easier solving problems which required a procedural knowledge of the arithmetic mean. In situations where the arithmetic means had been calculated wrongly, undergraduate students in Mevarech's (1983) study experienced difficulty in identifying them.

In a study by Prodromou (2013), she also observed that both students and teachers had misconceptions with the language surrounding the arithmetic mean. They were seen to be using the arithmetic mean and average interchangeably just as have been observed in other studies (Moore, 2007; Armah & Asiedu-Addo, 2014). An average or measure of central tendency are terms used in statistics to describe a typical or a representative value which denotes the center, middle, or expected value of a larger set. The most common measures of central tendency encountered in any introductory statistics course are the arithmetic mean, median and the mode. Computing any of these to appropriately represent a data set will depend on the type of data. Meaning, one of these measures will always be more appropriate than the others for any set of data. However, according to Upton and Cook (as cited in Groth & Bergner, 2006), some authors also describe the arithmetic mean as the average of a data set, while others describe it as the only type of average. Groth and Bergner (2006) also found that the most common explanation pre-service teachers gave to the arithmetic mean was to equate it to average.

It must however be noted that there are other specialized measures of center. Some of which are: the harmonic mean, used for finding “average per”, eg. speed, and the geometric mean, used for finding averages of percentages, ratios, indexes, and growth rates (Marnich, (2008). It is thus important for teachers to teach students to be aware of these other types of averages so that they will know the one to use at any point in time and not only take the arithmetic mean as the only average.

These and other misconceptions, Prodromou (2013) stated, have been born out of poor conceptual understanding from an algorithm-centric school system. She therefore stated that students appear to have a shallow understanding of the concept of the arithmetic mean when they are refused the chance to investigate the conceptual meaning of the arithmetic mean but instead memorize the algorithm. When Cai (1998) explored the conceptual understanding of the averaging algorithm of 6th grade students, he observed five main error types. Some of these included: minor omissions, the use of the guess and check method, unjustified symbol manipulation and incorrect use of the computational algorithm. Out of these, he realized that the largest percentage of the students committed the latter error. That is, though the students used the averaging algorithm (add them up and divide), they did not apply it correctly in the problem situations encountered.

It is easily observed that the moment the arithmetic mean is presented to students, they tend to use its algorithm as it is. As to whether they really understand it or not (Groth & Bergner, 2006) is not an important issue to them. And the rote algorithmic instruction as well as the early introduction of it to students, according to Marnich (2008), may cause a short circuit in the student’s reasoning.

Other studies have reported that students often make poor choices when asked to select a measure of center that will best describe a data set (Zawojewski &

Shaugnessy, 2000). On a test item with a structured data set in which the median was a better measure of center than the arithmetic mean, only 4% of the Grade12 students correctly used the median as a summary measure and also explained why it was the suitable measure to use. Callingham (1997) and Groth (2002) observed similar situations when the former gave a similar structured item to some pre-service and in-service teachers and the latter with students enrolled in an Advanced Placement high school statistics course. It is therefore clear that the arithmetic mean, as well as the weighted mean is not a simple computational algorithm and that it is not well understood (Pollatsek, Lima, & Well, 1981; Gattuso & Mary, 1998).

In a study by Garcia Cruz and Garrett (2008), results obtained showed that students, in their calculation of the arithmetic mean were not conversant with atypical or extreme values, and as a result had no idea what to do with them when encountered in a set of data. Even though the properties of the arithmetic mean have existed in the literature for a long time, a majority of both teachers and students consciously or unconsciously are not aware of them. Generally, the results of Garcia Cruz and Garrett's (2008) study revealed that the students were not conversant with some of these properties of the arithmetic mean, notwithstanding its fundamental nature. One thing that was worth noting in the study was that both the secondary school students and university students used as participants did not show any significant differences regarding the levels of interpretation given to the arithmetic mean.

Even though the APOS Theory has been used to analyze the understanding of students in a number of statistical and mathematical concepts (e.g. Maharaj, 2010; Hatfield, 2013), only Mathews and Clark (2007) and Clark, Kraut, Mathews and Wimbish (2007), to the best of my knowledge, have used it to study students' conception of the arithmetic mean. In Mathews and Clark's (2007) study, all students interviewed had moved beyond an action conception of the arithmetic mean. They had internalized the notion of inputting numbers, computation taking place, and a number as an output. They all had the computational algorithm for the arithmetic mean under their internal control, hence indicating a process conception of the arithmetic mean, however, when probed further on what the arithmetic mean represents, they returned to the “add them up and divide algorithm.” These students did not consider the arithmetic mean, a measure of central tendency, as an object with properties, or a cognitive entity. According to them, it is only a process.

The study by Clark, Kraut, Mathews and Wimbish (2007) was a replication of Mathews and Clark’s (2007) but with some modifications: the population was extended to four campuses instead of one in the former, participants were no longer restricted to fresh students, and the sampling procedure was also modified. Results also showed that none of the seventeen students used as participants was limited to an action conception of the arithmetic mean. All participants had interiorized the “add them up and divide” algorithm into a process. However, three different trends were observed beyond this level. Three of the students were limited to the process conception of the arithmetic mean. Some had formed a weak object conception that appeared to coexist with a predominately process concept image. It was interesting to note that only two of the seventeen university students used as participants had formed a “robust, rich object conception of the arithmetic mean that is compatible with the understanding that mathematicians and statisticians would expect.

2.7 Studies in Students' Conception of Variation

Statisticians and statistics educators have acknowledged variation to be the core of statistics (Orta & Sanchez 2011). Garfield and Ben-Zvi (2005) identified variability to be at the heart of statistics, and a major component of statistical thinking (Pfannkuch, 1997). The variability of a data set can be examined through its distribution which Wild (2006) stated, functions as a lens. In Pfannkuch's (2008) study, Ray, a statistician identified statistics to be basically the science of variation and statistical thinking to be mainly about evaluating variation.

According to Snee (1999), "if there was no variation, there would be no need for statistics and statisticians" (p. 257). Variation occurs everywhere in our human experience. Different results are obtained when different people measure the same quantity with the same instrument (Moore, 1990) or even when an individual measures the same quantity with the same instrument at different times. Cobb and Moore (1997) claim that what gives statistics a particular content and sets it apart from mathematics is the emphasis on variation. Hence, understanding the concept of variability or spread of data is a major factor in understanding distribution, and this is necessary for statistical inferences (Garfield & Ben-Zvi, 2008).

In Cobb's 1992 report which served as the basis for the GAISE college report, it was recommended that to emphasize statistical thinking, teachers were to help students recognize the ubiquitous nature of variability which is the essence of statistics as a discipline. Every introductory statistics course is aimed at producing statistically educated students. This means students must develop statistical literacy and the ability to think statistically. To achieve this, a number of goals were enumerated in the GAISE College report among which was that students must be taught to be certain of the fact and also understand why variability, the essence of statistics, is natural or ever present, quantifiable and explainable. Though the statistical techniques to help achieve these goals were important, the knowledge students gain after going through these techniques was more important (Aliaga et al., 2005).

Even at the lower levels of education, k–12, it is recommended that teachers assist students to understand the nature and sources of variability since statistical problem solving and decision making depend on an individual's ability to understand, explain, and quantify the variability in a data. This is because it is this focus on variability in data that sets apart statistics from mathematics (Franklin et al, 2005).

In a study of Turkish 8^{th}grade students' statistical literacy of average and variation,Yolcu and Haser (2013) found that although the majority of participants were able to explain the concept of variation in various contexts, their responses in other contexts indicated they considered there were more variation where the data set consisted of values of the same numbers. These responses showed a sign of possible misconception about the concept of variation of the students.

Shaughnessy (1997) observed a lesser attention given to research on variability. Giving reasons to this Shaughnessy (1997) indicated "statisticians have traditionally been very enamored with the standard deviation as the measure of spread or variability, and teachers and curriculum developers often avoided dealing with spread because they felt they could not do so without introducing standard deviation" (p. 11). The standard deviation, he noted, is computationally complex and difficult to motivate, particularly with beginning students. Another reason for this lack of attention he cited was that people are comfortable using measures of centers or averages for predicting into the future or for comparing groups, even though they are not always used correctly. When spread or variation is used in these predicting or comparing processes it only creates confusion in people's ability to make clean predictions or comparisons. Lastly, he indicated that the concept of variability may neither be within the comfort zone of many people nor their zone of belief.

Since descriptive statistics are mostly used in our everyday activities, it is necessary to understand basic statistical concepts in descriptive statistics such as the standard deviation. The standard deviation as a measure of variability is another fundamental concept taught in every introductory statistics class. It is the commonly used among the measures of variation and the commonly reported in statistical reports. Orta and Sanchez (2011) described it as the most suitable measure of variability, but one students find difficult to comprehend. Al-Saleh and Yousif (2009) identified it as a vague concept. Unlike other summary statistics, the standard deviation is one concept not fully understood by students. Most students in introductory statistics courses, though can compute the standard deviation for a data set, do not comprehend its value and importance (Al-Saleh & Yousif, 2009; Garfield & Ben-Zvi, 2008). Nevertheless, delMas and Liu (2005) indicated that an incomplete understanding of the standard deviation will affect students' understanding of other advanced and complex concepts like sampling distributions, inference, and p-values.

Turegun (2011) recounted his situation when he started teaching an introductory statistics course. He confessed he had not developed any conceptual understanding of the statistical topics himself. He had little or no difficulty performing calculations using his procedural understanding of the formulae and recipes, but could not give an explanation nor describe the ideas behind those formulae and procedures. Turegun (2011) found similar inconsistencies between conceptual and procedural understanding of introductory statistics topics among his students. One of such concepts was the standard deviation. Using the algorithm he, as well as his students, could calculate the standard deviation with their calculators once they knew which keys to use. However, explaining the concept of the standard deviation in the context of the particular data set was problematic.

Reading & Shaughnessy (2004) attributed part of students' difficulties in working with the standard deviation to the lack of accessible models and metaphors for students' conceptions of the concept. delMas and Liu (2005) also observed that most teachers tend to stress on the teaching of the formula of the standard deviation and practicing it with performing calculations. The authors also stated that teachers link the standard deviation to the empirical rule of the normal distribution, thereby, depriving students of its actual conceptual understanding. According to Delmas and Liu (2005), a conceptual model of the standard deviation is required to develop an instructional procedure that helps students to understand the concept. They claimed that a model of that nature should involve the coordination of numerous fundamental statistical concepts out of which the concept of standard deviation is built. Two of which, to them, are the concepts of distribution and the arithmetic mean.

In the study of Matthew and Clark (2007) which was to examine the conceptions of mean, standard deviation and the Central Limit Theorem most successful students had immediately after an introductory statistics course, their results showed that students did not have a grounded conception of the standard deviation. Out of the eight students who were interviewed, it was found that three of them had an action conception of the standard deviation, the lowest level of understanding according to the APOS Theory. At this level, the students only saw the standard deviation just as a rule or a formula to be followed, and they were not able to describe the algorithm for finding the standard deviation.

Contrary to the case of the arithmetic mean, not one of the eight students demonstrated an appropriate process conception of the standard deviation. Matthew and Clark (2007) therefore attributed this to the fact that most of the students had not been exposed to the concept of the standard deviation before the elementary probability and statistics course in their first semester at the university level. Moreover, the algorithms used for the computation of the standard deviation were evidently more complex, and their use was groundless in the minds of the students. The lack of a correct process conception of the standard deviation exhibited by the students indicated that the standard pedagogical treatment of this topic for these students was ineffective (Matthew and Clark, 2007).

Even though Clark, Kraut, Mathews, and Wimbish (2007) replicated the study of Matthew and Clark (2007) with some modifications, the results were similar. Their results also indicated that a third of the students used in this study only demonstrated to have a partial action conception of the standard deviation. This is because in some cases, some were unable to calculate or describe the standard deviation. Of those who had even progressed past the action conception of standard deviation, they seemed to also have a limited process conception of the standard deviation. While some of the participants had a relational understanding of the procedure for computing the standard deviation, others demonstrated they have only an instrumental understanding of this process. Those who understood the process relationally, found the successive distances of the measurements from the mean of the data, while those who understood it instrumentally were involved in merely subtracting and squaring the numbers with no cognitive link to the concept of distance. In any case, even though students will interiorize the procedure for calculating the standard deviation into a process, it will be done in an abstract sense by imputing data, performing calculations and yielding an output.

2.8 Summary

The review points to the fact that most students lack the conceptual understanding of the concept of the arithmetic mean and the standard deviation. Though students are able to do computations on these statistical concepts, they are not able to explain their results in connection to the set of data from which the answers were obtained. Statisticians and statistics educators have attributed this to the traditional lecture method and instructional strategies used by teachers in introductory statistics courses. These have not been effective in developing the conceptual understanding of statistical concepts and reasoning abilities of students and thereby not promoting statistical literacy among them (Gal, 2003; Garfield, Hogg, Schau, & Whittinghill, 2002; Hassad, 2008).

The statistics education research and reform have supported efforts to change teaching practices to consider the importance of developing students' conceptual understanding other than just a focus on mechanical calculations (Chance & Garfield, 2002). The traditional methods of learning whereby students passively listen to lectures and work in isolation have been the causes for rote memorization and learning of fragmented facts (Meletiou-Mavrotheris & Lee, 2002) leading to instrumental learning. In such a teaching environment, research has shown that students do not learn what their teachers expect of them (Chance, delMas, & Garfield, 2004; delMas & Liu, 2005) and these challenges encountered by student consequently lead to misconceptions in statistical reasoning (Chan, & Ismail, 2013).

The review is thus in the direction of the problem of the study. This will help to compare results of the current study, from the Ghanaian setting, to the majority of studies conducted outside Ghana into students' conceptual understanding of the arithmetic mean and the standard deviation.

CHAPTER THREE

RESEARCH METHODOLOGY

3.0 Overview

This chapter discusses the research design and the researcher's subjectivity statement.Thepopulation, the sample, and the sampling technique used are also described. The research instruments used in the data collection, as well as the procedure used to collect data are all discussed. The chapter also presents how the scoring of the test and data analysis were done.

3.1 Research Design

Every empirical study has either an implicit or an explicit, research design (Yin, 1994). This design, in the most basic sense, is the logical sequence that links an empirical data to the initial research questions of a study and, eventually, to its conclusions (Yin, 2009). The research design, according to Nachmias and Nachmias (1992), is:

> a plan that guides the investigator in the process of collecting, analyzing, and interpreting observations. It is a logical model of proof that allows the researcher to draw inferences concerning causal relations among the variables under investigation. The research design also defines the domain of generalizability, that is, whether the obtained interpretations can be generalized to a larger population or to different situations (pp. 77-78).

De Vaus (2001) on the other hand does not see the research design as just a workplan. To him, a work plan just details what must be done to bring a project to completion. Nevertheless,this work planmust be derived from the project's research design. He thus stated that the research design functions to ensure that evidence obtained will help the researcher to unambiguously answer his initial questions as possible.

In the current study, the researcher employed the exploratory case study design. To be more specific, the single case with embedded units was employed in this study. The nature of the research questions required data that was purely qualitative in nature, since the study was conducted to first, investigate the conceptual understanding first year mathematics students of UEW have about the arithmetic mean as a measure of central tendency and the standard deviation as a measure of variation at their entry stage. Then secondly, based on the conceptual understanding exhibited, find out the level of conceptual knowledge, according to the APOS theoretical framework, they have.This, being exploratory in nature, called for the exploratory case study design.

According to Creswell and Miller (2000) and Merriam (1998), research questions guide the selection of research methodology. When researchers want to discover the meaning people give to events they experience (Bogdan & Biklen, 2003; Denzin & Lincoln, 2003), the qualitative research methods are useful. Bogdan and Biklen (2007) described qualitative research as “an approach to social science research that emphasizes collecting descriptive data in natural settings, uses inductive thinking, and emphasizes understanding the subject's point of view” (p. 274). Glesne (2011) also defined qualitative research as “a type of research that focuses on qualities such as words or observations that are difficult to quantify and lend themselves to interpretation or deconstruction” (p. 283).

Creswell (2014) describes the qualitative research as an approach used to explore and understand the meaning people orgroups attribute to a social or human problem. This process of research, he said, involves evolving questionsand procedures. Data for this approach is typically collected at the participant’s location, data analysis is inductively built from particulars to general themes, and the researcher makes interpretations of the meaning of thedata(Creswell, 2014).

The qualitative research model has some basic characteristics. In the first place, it takes place in natural settings (Hatch, 2002; Marshall & Rossman, 2006), in the “real world” (Leedy & Ormrod,2005, p. 133).Data is usually collected in settings where the participants experience the problem or issue that is being studied (Hatch, 2002;Marshall & Rossman, 2006) without any interference. Also, the researcher is the key person in collecting data, (Hatch, 2002; Creswell, 2007), multiple data sources like interviews, observations and documents are employed to collect data, after which the researcher reviews it thoroughly and organizes them into themes. Inductive data analysis is used to build these themes in a “bottom-up” manner, into increasingly abstract units (LeCompte & Schensul, 1999; Marshall & Rossman, 2006; Creswell, 2009).

Further, in the qualitative paradigm, research concentrates on the views and meanings participants have about the issue under study not that of the researcher or what the literature says (LeCompte & Schensul, 1999; Hatch, 2002; Creswell, 2009). The initial research plan in the qualitative research paradigm is also not static. Any phase of the research process may be changed from the original study design as need be in the course of the study (Hatch, 2002; Marshall & Rossman, 2006). More so, the qualitative research paradigm is characterized by the use a theoretical lens. Studies are often viewed through the lens of a theoretical framework. It may also be centered on identifying a social or political context of the issue being studied (LeCompte & Schensul, 1999; Creswell, 2009).

The employment of interpretive inquiry is another characteristic of the qualitative paradigm. By this, researchers form an interpretation of what they see, hear, and understand. However, these interpretations cannot be disconnected from the researcher's background, past experiences, or prior understandings (Marshall & Rossman, 2006; Creswell, 2009). Lastly, the qualitative paradigm employs a holistic view of the social phenomena being examined. The researcher tries to create a complex picture of the subject under study by reporting multiple viewpoints, identifying key factors involved, as well as describing the "big picture" that evolves from the data. He is not restricted to a rigid cause-and-effect relationship between factors, rather, the researcher is free to identify and expand upon complex interactions in any given situation or setting (Hatch, 2002; Marshall & Rossman, 2006).

Leedy and Ormrod (2005), observed that qualitative research serves one or more of the following purposes:

- Description - revealing the nature of certain situations, settings, processes, relationships, systems or people.
- Interpretation - gaining insights into a particular phenomenon, developing new concepts or theoretical perspectives about the phenomenon and/or discovering problems that exist within the phenomenon.
- Verification - allowing the researcher to test the validity of certain assumptions, claims, theories, or generalizations in real-world contexts.

- Evaluation - providing a means through which a researcher can judge the effectiveness of particular policies, practices or innovations (p. 134).

According to Creswell (2007), a qualitative paradigm is employed when an issue or problem needs to be explored in great depth and detail or when the researcher wants to empower a group of individuals to share their experiences. This occurs when research questions begin with "how" or "what" (Patton, 2002; Seidman, 1998). According to Stake (1995), a qualitative research approach is justified when the nature of research questions require exploration.

After identifying the qualitative nature of the study, there was the need to indicate which design would be appropriate. Creswell(1998)outlined five designs in the qualitative research approach: Ethnographic research, Grounded theory, Phenomenological research, Narrative research and Case Studies. Ethnography, according to Harris and Johnson (2000), literally means a portrait of a people.They defined an ethnography as awritten description of a particular culture, comprising, the customs, beliefs, and behavior, which is based on information collected through fieldwork. Brewer (2000) also described ethnography as a qualitative study of people in their naturally occurring settings or 'fields' through procedures capturing their social meanings and ordinary activities, involving the direct participation of the researcher in the setting, if not also the activities, so as to collect data in a systematic manner without imposing meaning on them externally. In Ethnography, Creswell (2014) indicated that the investigator mainly collects data through observations and interviews to study the shared patterns of behaviors, language, and actions of an intact cultural group in its natural setting for a long period of time. LeCompte and Schensul (1999) describe the research process as flexible and evolving contextually in reaction to the lived realities the researcher comes across in the field setting.

Groundedtheory wasdescribed by Strauss and Corbin(1990)as a methodology for developing theory from data that has been systematically gathered and analyzed through the research process. From sociology, grounded theory is a qualitative design of inquiry within which the researcher develops a general, abstract theory of a process, action, or an interaction that is rooted in the opinions of participants in a study (Creswell, 2014). The two main characteristics of this design as pointed out by Creswell (2003) are "the constant comparison of data with emerging categories and theoretical sampling of different groups to maximize the similaritiesand the differences of information" (p. 14).

Phenomenology was also defined by Hancock (1998) literally as the study of a phenomenon. A way of describing something as being part of the world we live in. These phenomena, according to Hancock (1998) may be events, situations, experiences or concepts. Therefore Donalek (2004) opined that phenomenological studiesexamine human experiences through the descriptions given by the individuals involved in the study. As a research design from philosophy and psychology, Creswell (2014) describes phenomenological researchas a designinwhich the researcher describes the lived experiences of individuals about a phenomenon as described by participants (Donalek, 2004; Hatch, 2002) and the researcher tries to understand the essence of those experiences (Hatch, 2002). The design has strong philosophical underpinnings andtypically involves conducting interviews (Giorgi, 2009; Moustakas, 1994; Donalek, 2004).

According to Moen (2006), "a narrative is a story that tells a sequence of events that is significant for the narrator or her or his audience" (p. 4). Thus

Gudmundsdottir, cited in Moen (2006) defined narrative research as the study ofhuman beings' experience with the world, as they are collected and narratives of these experiences written by narrative researchers. As a design from the humanities, Creswell (2003) also describes itas "a form of inquiry in which the researcher studiesthe lives of individuals and asks one or more individuals to providestories about their lives" (p. 15). According to Clandinin and Connelly as cited by Creswell (2014), the information as received by the researcher is then retold or restoried into a narrative chronology and a collaborative one which combines opinions from the life of the participant with the researcher's own.

Case studies, the last of the qualitative approaches listed by Creswell (1998; 2014), are a research design in which a researcher carries out an in-depth analysis of a program,an event, activity, a process, or one or more individuals referred to as a case or cases (Creswell, 1998).Rose, Spinks, and Canhoto (2014) indicated that the word 'case' means 'an instance of' and what characterizes a case study research design is the investigation into one or more specific 'instances of', what comprise the cases in the study. These cases, according to Yin (2009; 2012) are bounded by time and activity, and researchers use a number of procedures to collect detailed information over a continuous period of time whilst 'retaining the holistic and meaningful characteristics of real-life events' (p. 4).

As indicated by Yin (1994), the case study design is "an empirical inquiry that investigates a contemporary phenomenon within its real life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13). "As a research method, the case study is used in many situations, to contribute to our knowledge of individual, group, organizational, social, political, and related phenomena" (Yin, 2009, p. 4). Zainal (2007) also posits that the role case studies play in research becomes more prominent when discussing issues regarding education, sociology and community based problems like poverty, unemployment, drug addiction, illiteracy, etc. (Zainal, 2007). Thus, the phenomenon investigated in the current study is the 'conceptual understanding of the arithmetic mean and the standard deviation' and the case being 'the first year students of U. E. W.

Yin (1994) further pointed out that a case study can either be explanatory, descriptive or exploratory depending on the type of research questions to be answered, the degree of control over actual behavioral events, and the amount of focus on contemporary events as against historical events. Case studies are explanatory when they answer 'how' or 'why' questions with little control by the researcher over occurrence of events. They concentrate on phenomena in real-life contexts and do a thorough examination of data at both the surface and deep level so as to give an explanation to the phenomena in the data. "Explanatory cases are also deployed for causal studies where pattern-matching can be used to investigate certain phenomena in very complex andmultivariate cases"(Zainal, 2007, p. 3).

'Who' and 'where' questions with their derivatives 'how many' and 'how much' are more descriptive case studies with no control by the researcher (Yin, 1994). Descriptive case studies generally try to describe culture or sub-culture, and also try to find out the key phenomena (Dudovskiy, n. d.). They describea phenomenon in its natural setting occurring in a data in question (Zainal, 2007). Their aim is to analyze sequences of interpersonal events after a period of time (Dudovskiy, n. d. ; Zainal, 2007).

Exploratory case studies on the other hand find answers to 'what' or 'who' questions (Yin, 1994). A number of data collection methods such as interviews, questionnaires, experiments, etc. are used in collecting data in *e*xploratory case studies (Dudovskiy, n. d.). They are often used to investigate a phenomenon in a data which is of interest to a researcher. The purpose of the general questions asked is to give room for further investigation into the observed phenomenon (Zainal, 2007). The current study was found to be exploratory in nature.

Apart from identifying the case or cases in a case study and the particular type of a case study design to be used, Gustafsson (2017) posits that it is imperative for the researcher to decide whether it is appropriate to use a single or multiple case study to help understand the phenomenon under study. Gustafsson (2017) further described that if a study includes more than one single case, then the multiple case study has been employed. Consequently, aside being exploratory, the current study was also a single case study, since the researcher considered only one case: first year students admitted into the Mathematics Education Department of the University of Education, Winneba. According to Dyer & Wilkins (1991), the single case study, when employed helps researchers to come out with new theoretical relationships and be able to question old ones leading to deeper and better understanding of the phenomenon under study than multiple case which causes the researcher to have less contextual insight into what he can communicate.

Further, since the researcher used three different year groups of students admitted into the Department of Mathematics Education of the University of Education, Winneba: 2012/13, 2013/14 and 2014/15, the single case study with embedded units was adopted. According to (Yin, 2003), this type of case study gives the researcher the ability to look atsubunits that are located within a larger case. This, the researcher did to investigate the differences and similarities in the conceptual understanding of the arithmetic mean and the standard deviation exhibited by these three year groups of students.

3.2 Subjectivity Statement

In qualitative methodology, there is the recognition that the researcher's subjectivity is closely involved in scientific research (Ratner, 2002). This subjectivity guides the research process from choosing a topic, through to formulating hypotheses, to selecting the data collection procedures to data interpretation (Glesne & Peshkin, 1992; Ratner, 2002). The researcher in a qualitative study is thus encouraged to think of the ideals and purposes he conveys to his research and how they affect the research process. Other researchers are also to reflect on these ideals that any researcher employs (Ratner, 2002).

I trained as a professional teacher specializing in mathematics as a result of my love for the subject. Having taught core mathematics for three years in the JHS and also two years in the SHS as both a core and elective mathematics teacher, I have a good impression of what constitute a good teaching and learning process and environment. During these years, other mathematics teachers teaching the subject at these levels were also observed. As I continued my studies in mathematics education at the tertiary level, my thinking and experiences were modified as I learnt more about psychology of learning mathematics and other mathematics methodology courses. This gave me more insight into the teaching of mathematics at both the JHS and the SHS levels.

Joining the Department of Mathematics Education of the University of Education, Winneba as a lecturer added to my experiences. It is the mission of this university to train competent teachers for all levels of education as well as conduct research, disseminate knowledge and contribute to educational policy and development. As a result, I have been exposed to supervising pre-service or practicing teachers in mathematics classrooms. This has again equipped me with both theoretical and practical knowledge as well as teaching standards to use as a basis for analyzing what helps students to build conceptual knowledge of mathematical concepts. I also had the opportunity to read and conduct more research in the field of mathematics education which broadened my knowledge in the educational field, especially in the teaching and learning of introductory statistics, a course I have taught for about a decade.

Though I have taught other courses in both mathematics methodology and content in the university, coming into contact with various researches in statistical literacy, reasoning and thinking has given me further insight into what must go on in an introductory statistics class or lesson to facilitate conceptual understanding of concepts. What heightened my interest was the information I learnt from the ARTIST website which included the kind of questions that encourages statistical literacy, reasoning and thinking. These added to my commitment to teaching for understanding and student centered education in a transformational, all-inclusive approach which has been part of a philosophy I have held throughout my teaching.

During these years of teaching at the university, I have come into contact with students who have demonstrated only procedural understanding and knowledge of statistical concepts. Since the arithmetic mean and the standard deviation are basic statistical concepts which are encountered in everyday life, students in this information age must be able to demonstrate understanding and use them appropriately. This is the reason why it has been included in the Ghanaian mathematics syllabi from the primary to the SHS level, with specific objectives to help students interpret these concepts.

These experiences helped me to know the kind of items to include in the test, the kind of questions to ask during the interview and generally what to do throughout the study. In effect, both the data collection and its analysis were done by me. Ensuring the study was conducted in the natural setting, I made sure the students' conceptual understanding of the mean and the standard deviation was not interfered. The knowledge they have is as a result of their mathematical experiences at both the JHS and the SHS levels. The test was given to them before formal lectures. Immediately afterwards, the interviews were conducted, also by myself, to ensure the right results were obtained.

Stake (1995) describes the role of the researcher in case studies, as a teacher, advocate, evaluator, biographer, interpreter, in constructivism and relativity. Though these roles may change for specific situations, he stated that the researcher cannot be separated from the research. The researcher, he further explained, "is the agent of new interpretations, new knowledge, but also new illusion" (p. 99). Therefore, this research, though value-laden, is not free from biases (Creswell, 2007).

3.3 Population

The population for the study consisted of all first year students admitted into the Department of Mathematics Education of the UEW. The study involved the cohort of students admitted on the basis of their performance in six subjects (including core and elective mathematics) in the West African Senior School Certificate Examination (WASSCE). In terms of their mathematical ability, students who offer mathematics in the various tertiary institutions can be said to possess good or above average knowledge in the subject.

The study targeted this population because the researcher, in her ten years' experience of teaching such cohort of students, had observed that majority of such senior high school graduates who possess good or above average knowledge in the subject enter the mathematics education programme without having a good grasp of the concepts of the arithmetic mean and the standard deviation. In fact, this has implication for the greater majority of senior high school graduates who enter other programmes that require little mathematics as well as those who fail to obtain credit passes in the subject.

The study, therefore, targeted students admitted into the Department of Mathematics Education of the UEW, in three academic years: 2014/2015, 2013/2014, and 2012/2013. Table 3.1 shows the population of students admitted to the department in the three years.

Table 3. 1

Distribution of Students Admitted to the Department of Mathematics Education UEW for 2014/2015, 2013/2014, and 2012/2013 Academic Years

	Male		Female		Total
Year Groups	**N**	**%**	**N**	**%**	**N**
2014/2015	202	88.0	28	12.0	230
2013/2014	204	92.0	18	8.0	222
2012/2013	148	94.0	10	6.0	158
Average	**185**	**91.0**	**18**	**9.0**	**203**

Source: (UEW Publications Unit, 2016)

3.4 Sample and Sampling Techniques

The purposive sampling technique, to be precise, the homogeneous type, was used in selecting students from the population described because all of them have gone through the Ghanaian JHS and SHS mathematics curriculum as well as the SHS elective mathematics curriculum which happened to be the baseline for the study. The purposive sampling technique was deemed appropriate since the emphasis was on quality instead of quantity and the main aim was not to capitalize on numbers, but to become "saturated" with information on the topic (Padgett, 1998, p. 52).

The convenience sampling technique was used to select the samples from the year groups. The reason being that, for each of the year groups, students who had reported at the time of administering the test were used as the sample. The researcher planned to use at least 60 percent of the students in each year cohort. However, in 2012/2013 academic year as high as 88.0% of students admitted were involved since only 158 students were enrolled. In the 2013/2014 academic year 63.0% were involved and in the 2014/2015 academic year 65.0% were involved. In the three academic years, 2012/13, 2013/14 and 2014/15, the number of students sampled into each group were 140, 140 and 150 respectively (see Table 3.2). A letter introducing students to the objectives of the study was given to students and their consent sought before partaking in the study. Samples of the letter and consent form are presented in Appendices A and B respectively.

The simple random sampling technique was used to select ten students from the 2014/15 academic year batch to be interviewed. This was to assist in obtaining more information on the conceptual understanding of the students on the concepts under study.

Table 3. 2

Proportion of Students Sampled for the Study in the Three Academic Years

Year Group	Number Admitted	Number Sampled	Percent Sampled
2012/2013	158	140	88.0
2013/2014	222	140	63.0
2014/2015	230	150	65.0

3.5 Instruments for Collecting Data

Two instruments were used for collecting the data for the study: a test and an interview guide.

3.5.1 Test

The test consisted of two parts. The first part was designed to obtain the demographic characteristics of the participants in the study. The second part contained items meant to investigate the conceptual knowledge of the arithmeticmean and the standard deviation that students have constructed as a result of the learning experiences they have undergone through the Ghanaian JHS and SHS core mathematics and the SHS elective mathematics curricula. (See Appendix C for sample of test). Even though participants were asked to indicate their school identification numbers, they were made to understand that the exercise was not going to be part of their continuous assessment. This was done just for the purpose of identification and triangulation. It was not the intention of the researcher to analyze the test results quantitatively.

There were ten (10) test items in the second part of the test, with one having sub items on both the arithmetic mean and the standard deviation. Five (5) were solely on the arithmetic mean and four (4) also on the standard deviation. One of the test items (item 15) which was on the concept of the fair share of the arithmetic mean was adapted from Marnich's (2008) pilot study items. Items 12, 13 and 17 were constructed by the researcher. The other items were either retrieved from some statistics test items online or from the assessment builder section of the Assessment Resources Tools for Improving Statistical Thinking (ARTIST) website. The ARTIST, which is a National Science Foundation (NSF)-funded Web project has developed a web-based assessment resource for introductory statistics courses. The goal is to help teachers in assessing statistical literacy, statistical reasoning, and statistical thinking in first courses of statistics.

Few of the items obtained online were adapted, particularly names were changed to reflect our Ghanaian background. In all, there was one multiple-choice item. However, participants were given the chance to give reasons for their choice of answer. Apart from the multiple-choice items, all other items were constructed response type where spaces were provided for participants to provide their own answers and give explanations for the answers. Instead of focusing on computations and procedures, the test items focused on statistical reasoning and conceptual understanding of the concepts of the arithmetic mean and the standard deviation.

Approval for the use of the test was given by my supervisors after it was piloted on fifty (50) first year undergraduate mathematics students of University of Cape Coast (UCC) for the 2012/13 academic year. This was done to ensure that participants do not only understand the test items, but also they understand them in the same way. After the pilot testing, and with the assistance of my supervisors, items that seemed to be ambiguous to students were either deleted or re-framed.

In a qualitative study, one way of achieving credibility is through the triangulation of data. This allows a researcher to compare numerous types and sources of data obtained throughout a study to help achieve consistency of the results. According to Scriven (1991), triangulation is, “the attempt to get a fix on a phenomenon or measurement by approaching it via several independent routes” (pp. 364- 365). Patton (1987) explained triangulation as:

> Comparing observational data with interview data; it means comparing what people say in public with what they say in private; it means checking the consistency of what people say over time; and it means comparing the perspectives of people with different points of view. It means validating information obtained through interviews by checking program documents and other written evidence that can corroborate what interview respondents report (p. 161).

As a result of the identification numbers used by respondents, explanations to responses of test items were compared with students' responses to questions in the interview. Also, what students wrote down in their effort to explain issues during the interview were also compared with their responses to help interpret what participants wanted to put across. This helped to ensure data triangulation.

3.5.2 Interview Guide

Since the case study involves an in-depth study, interviews are also used to collect data. The semi-structured interview was used by the researcher to gather data on only the 2014/15 year group, on their conceptual understanding of the arithmetic mean and the standard deviation. Apart from the specific questions to be asked during a semi structured interview, the interviewer is allowed the flexibility to modify the sequence of questions or to probe for more in-depth responses as considered appropriate from the answers and comments provided by the participants (Merriam, 1998). Though other open-ended questions may be asked as well, the two broad, questions are meant to focus or direct the data collection procedures in order that the researcher may eventually understand the common experiences of the participants.

This interview is likened to a clinical interview, which is a dialogue between an expert, like a psychologist or a physician, and a client. It is designed to help the expert diagnose and plan treatment for the client. It is thus a conversation with a purpose, as well as clearly defined roles. Here the expert is in charge and asks more questions. It is worth noting that the interview is really only about the client. Finally, a clinical interview occurs within a defined time frame, unlike a friendly conversation which can start and end at any time.

All interviews were conducted in the office of the researcher. The researcher made sure each interviewee was given a brief recap for the purpose of the study as well as the instructions as to what to do at the onset of the interview. During the interview, interviewees were made to feel at ease with some familiarization questions at the initial stage. During the familiarization period, there were questions which sought to get participant's demographic characteristics. This was done in order to compare demographic information gathered during the interview with that gathered during the test. However, the main questions each of the nine interviewees answered were:

- What is meant by the term “arithmetic mean" in statistics?
- What is meant by the term “standard deviation" in statistics?

A copy of the interview protocol can be seen in Appendix D. Interviewees were given the chance to answer each of the questions without prompting.

Nonetheless, depending on their responses, the interviewer provided clues, asked for clarifications, and asked follow up questions. This continued until the researcher was sure no more information was forthcoming. This meant the interviewee had expressed fully the knowledge he or she possessed about these two concepts.

During the interview, students were provided with sheets of papers to write or illustrate anything that would help them in explaining themselves better. They were also asked to give examples where they could. The interviewer made sure she maintained a relaxed environment to ensure students felt free to express themselves. Each of the interviewees was scheduled to spend thirty minutes during the interview. Nevertheless, some exhausted all what they had to say before the scheduled time.

All the interviews were audio-taped and then transcriptions of these sessions were produced to complement the record of written work which the student completed during the interview as well as the test item. This was made easier as a result of the student's identification numbers used. The transcripts and written work were read cautiously and analyzed in order to produce a list of mathematical issues that showed up during the interviews. Concentrating on these issues, results were obtained about the mental constructions that students appear to have formed on the arithmetic mean and the standard deviation.

3.6 Issues of Trustworthiness

Trustworthiness is a way for qualitative researchers to control possible sources of biasedness in a study's design, its implementation, analysis, and interpretation that matches the notions of internal validity, external validity, reliability, and objectivity from more conventional, scientific studies (Lincoln & Guba, 1986). Thus, this issue of trustworthiness, quality, rigor or the soundness of any qualitative research work cannot be overlooked. Whereas quantitative researchers look at the reliability, objectivity and validity (i.e. internal and external) to ensure the trustworthiness of their findings, qualitative researchers look at the credibility, dependability, transferability and confirmability as criteria for trustworthiness (Guba, 1981; Schwandt, Lincoln, & Guba, 2007). These four criteria: credibility, dependability, transferability and confirmability, are discussed in the design and implementation of this current study.

3.6.1 Credibility (Internal Validity)

Credibility, which is parallel to internal validity (Guba & Lincoln, 1989), seeks to ensure that research work measures what it intends to measure. According to Merriam (1995), it asks the question, "How congruent are the findings of a study with reality"? (p. 53) The credibility criterion includes establishing that the findings of a qualitative research are believable from the point of view of the participants in the research study. Since from this point of view, qualitative research is meant to provide a description or an understanding of the phenomena of interest from the participant's point of view, the participants are the only ones who can legitimately judge whether the results are credible (McAninch, 2015). In ensuring credibility, researchers try to establish the fact that a true picture of the phenomenon under study is presented (Shenton, 2004). This addresses the issue of internal validity as in a quantitative study.

Before information was obtained from participants of this study, they were briefed on the purpose of the study and also made to feel at ease. They were made to understand that results were not going to be part of their continuous assessment. They were also not restricted to any time duration during the test. This was to ensure they had enough time to think and express themselves. Similarly, during the interview, participants were given enough time to express themselves until the researcher realized they had exhausted all they knew about the topics under study.

In ensuring triangulation, the study adopted two different data collection strategies: a test and an interview. This, according to Guba (1981) and Brewer and Hunter (1989) ensures that individual limitations of participants' are catered for and their respective benefits exploited. The use of these strategies created an avenue where supporting data were obtained to help clarify responses, attitudes and behaviour of participants' which were not clear. This was easily done as a result of participants' identification numbers used.

The interview questions were slightly modified questions from the test which gave respondents the opportunity to clarify their responses in the test. This gave chance for further probing into issues raised by respondents. After the interview, respondents' views were summarized for them to affirm whether they match with what they really wanted to put across.

3.6.2 Transferability (External Validity)

Transferability describes the extent to which results of a qualitative study can be transferred to other settings with different respondents (Bitsch, 2005). It represents how the findings of a study may be applicable to other situations. Researchers are particularly concerned with the extent to which the "results of the work at hand can be applied to a wider population" (Shenton, 2004). Guba & Lincoln (1982) have described it to be parallel to external validity or generalizability in quantitative studies.

To facilitate transferability, Bitsch (2005) opines that the researcher must provide a thick or a detailed description of the study and also use purposive sampling in selecting respondents. By thick description Anney (2014) explains, it involves the researcher clarifying all the research processes from data collection, context of the study to production of the final report. According to Shenton (2004), providing adequate detail of the context of the study helps a reader to decide whether the existing environment is comparable to another situation he or she is familiar with. It also helps other researchers to replicate the study with similar conditions in different settings (Anney, 2014; Guba, 1981).

Teddlie and Yu (2007) defined purposive sampling as “selecting units (e.g., individuals, groups of individuals, or institutions) based on specific purposes associated with answering a research study’s questions” (p. 77). It requires that one considers the characteristics of the members of a sample as those characteristics are related directly to the research questions (DeVault, 2017). When used, purposive sampling helps the researcher to concentrate on key respondents, particularly knowledgeable on the issues under study (Schutt, 2006). According to Cohen, Manion, and Morrison (2011), in-depth findings are obtained when purposive sampling is used than in probability sampling methods. It also helps the researcher in his decision as to why a specific group of respondents must be used in a study (Bernard, 2000).

In the current study, both strategies have been employed to ensure transferability. The study employed the use of the purposive sampling technique to ensure specific and varied information is emphasized than a generalized and aggregate information, which would have been the case in a quantitative research. Also a detailed description of the processes used in the study, as well as the results have been spelled out to aid replication if need be.

3.6.3 Dependability (Reliability)

In parallel with the concept of reliability in quantitative studies, dependability discusses how stable the results of a study are over time. It answers the question: will the results of a study be the same when replicated with the same or similar respondents in a similar context (Bitsch, 2005)? McAninch posits that the question of dependability refers to the situation in which a different researcher repeats the same work, in the same setting, with the same methods and respondents and obtains similar results (McAninch, 2015). As such, in ensuring dependability of research findings, a researcher must provide evidence that if the study were to be replicated with the same or similar respondents in the same or a similar context, findings would be repeated. To do this, Bitsch (2005) proposes that a detailed and comprehensive documentation of the research process must be provided as well as every methodological decision. This, according to Shenton (2004) will help future researchers to be able to replicate the work.

For this study, the researcher made sure all research procedures were described in detail. Also the views of supervisors, advisors and experts in the field of qualitative research were sought to ensure the right research procedures were followed to confirm dependability.

3.6.4 Confirmability (Objectivity)

Comparable to researchers' concern of objectivity, the concept of confirmability is based on the fact that research is by no means objective (Morrow, 2005). It denotes the extent to which results of a study might be confirmed by different researchers (Baxter & Eyles, 1997). Confirmability describes the degree of impartiality or the extent to which the findings of a study are shaped by the participants and not the biasedness, motivation, or interest of the researcher. Bitsch (2005) posits that data, its interpretations, and findings must be based on individuals and situations apart from the researcher. Thus, according to Gasson (2004), confirmability concerns itself with the main issue that "findings should represent, as far as is (humanly) possible, the situation that is being researched instead of the beliefs, pet theories, or biases of the researcher" (p. 93). It is also "concerned with establishing that data and interpretations of the findings are not fabrications of the inquirer's imagination, but are clearly derived from the data" (Tobin & Begley, 2004, p.392).

Data triangulation was used as one of the strategies to ensure confirmability and to reduce the effect of biasedness of the researcher. Again, audit trail, which describes the detailed methodological or step by step description of how data was gathered and processed in this current study is given to ensure confirmability. Also, consultation with advisors during the collecting and processing of data helped to control biasedness on the part of the researcher.

3.7 Data Collection Procedure

Before formal lectures for the Introductory Probability and Statistics course began for each of these year groups under study, fresh undergraduates who had reported were organized to write the test. This was done to ensure that there is no interference with their previous knowledge, which was the knowledge gained based on their learning experiences from the JHS and SHS core mathematics as well as the SHS elective mathematics education.

On the 17th September, 2012, all fresh undergraduate mathematics students who had reported for the 2012/13 academic year were assembled at three lecture halls for the administration of the test. In the same way, the 2013/14 and the 2014/15 batches of participants took their tests on the 19th September, 2013 and the 24th September, 2014 respectively. All the three year groups took their tests around 3:00pm. Since the researcher wanted to give students enough time to answer the questions, no stipulated time was given for the tests. Students were to fully express their knowledge to their maximum satisfaction before they could present their answer scripts. The seating arrangement was done in such a way that students would do independent work and supervision was done by the researcher herself with the assistance of colleague lecturers from the Department of Mathematics Education of the UEW.

The interview was scheduled and took place on the 25th and 26th September, 2014 for only the 2014/15 year group in the researcher's office. The random sampling of those to be interviewed was done on the 24th September, 2014 after the administration of the test. The first five of the sampled participants for the interview were scheduled and informed of the interview the following day, which was the 25th

and the other five for the 26th of September. Before the interview, the test scripts of the interviewees were retrieved and studied to give insight into the responses they gave to the test items for further probing.

All the interviews were recorded and labeled with the index numbers of the participants as used in the test. The interviews had two main purposes. First, it was to collect more information about the students' mathematical experiences before formal lectures on the introductory statistics course. Then secondly, it was to investigate their conceptual knowledge of the arithmetic mean and the standard deviation. In doing this, the participants were thinking aloud. This helped in the analysis of the results to reveal how these students have conceptualized the concepts of the arithmetic mean and the standard deviation.

After the interview sessions, the recordings were saved onto the personal computer of the researcher till the time was due for analyses. In the transcriptions, on the recordings and in all other written work produced as part of this research study, participants were identified only by their registered index numbers.

3.8 Scoring of Test Items and Data Analysis

Scoring was done by the researcher. The study employed the quantitative content analysis. According to Marshall and Rossman (as cited in Trace, 2001), content analysis is "an overall approach, a method, and an analytic strategy" that "entails the systematic examination of forms of communication to document patterns objectively". It is used to determine the existence of words or concepts, existing in texts or sets of texts such as speeches, transcribed interviews, and published literature.

It aims at clarifying through close scrutiny, the content and language of these texts and how authors or respondents view and understand certain issues or phenomena.

According to "An Introduction to Content Analysis," (2004), in content analysis, researchers code the text or break them down into manageable categories on a number of levels which are word, word sense, phrase, sentence, or theme. They then quantify and analyze the presence, meanings and relationships of these texts and concepts, and make inferences about the main ideas in the texts. In quantitative content analysis, the occurrences of content units are tabulated (Franzoi, 2008).

The researcher was interested in whether a participants' response to an item was correct or wrong and the percentage of students getting the item right or wrong was noted. Furthermore, the responses of students to constructed response or open ended items were grouped into themes and percentages of students whose response fell in a theme were noted. The researcher also tried to find out if there was a match between participants' responses from the interviews and the test items. This was to assist in getting an in-depth understanding into what the participants were trying to put across.

In analyzing the data to answer research questions 1 and 3, the researcher looked out for the theme(s) majority of the participants used to describe the concepts of the arithmetic mean and the standard deviation. The theme(s) with the highest percentage(s) were considered as what participants have conceptualized the arithmetic mean and the standard deviation.

For research questions 2 and 4, where the researcher sought to determine the level of conceptual knowledge participants have according to the APOS Theoretical Framework, the researcher looked out for how participants described the concepts of the arithmetic mean and the standard deviation, the type of items they were able to answer and the knowledge needed to answer those items. This, the researcher did, without forgetting the characteristics expected of individuals at the levels of conceptual knowledge of the arithmetic mean as well as the standard deviation according to the APOS Theorem.

In this study, and according to the APOS Theoretical Framework, participants who were able to demonstrate competence in the computation of the arithmetic mean in the absence of a formula were considered to have an 'action' conception of the arithmetic mean. In the same way participants who were able to compute the standard deviation without external promptings, were considered to have an 'action' conception of the standard deviation.

For participants to demonstrate having a 'process' conception of the arithmetic mean, in the current study, they were to demonstrate the ability to be conversant with the algorithm of finding the arithmetic mean. They were to exhibit the ability to explain and describe the algorithm from memory without actually performing it. They were also to demonstrate the ability to reverse the algorithm of the arithmetic mean to find a missing number when the arithmetic mean of the data set is given. The same applied to the concept of the standard deviation. Participants were to be seen to explain and describe the algorithm for finding the standard deviation to have a 'process' conception of the standard deviation.

The current study considered participants who were able to explain and describe what the outcome of the algorithm for computing the arithmetic mean, and were able to identify and work within the properties of the arithmetic mean to possess an 'object' conception of the arithmetic mean. Mathews and Clark (2007), described a student who understands the standard deviation of a data set to be a measure of spread which roughly averages distances from the arithmetic mean to have an 'object' conception of standard deviation. Such a students must be able to distinguish between two data sets or graphical representations, which of them would have a higher standard deviation and also explain what a standard deviation of zero means. A student with an 'object' conception of the arithmetic mean would know and understand that the arithmetic mean of a data set must be within the extremes of the data set.

Finally, for participants to have a 'schema' conception of the arithmetic mean or the standard deviation, they were to demonstrate the ability to apply all the actions, processes and objects related to the concept and use them to solve problems on the concept and related concepts. During the interview, the researcher considered the properties of these concepts that participants were able to demonstrate in their descriptions of these two concepts and linked it to the characteristics that must be portrayed by an individual at each of the levels of conceptual knowledge according to the APOS Theorem. This, in conjunction with the responses to the test items helped her to place participants at the level of conceptual knowledge of the arithmetic mean and also for the standard deviation.

3.9 Summary

In this chapter, the researcher described the design of the study: the exploratory case study design of the qualitative approach. The population, as well as the sample was described. In all, a total of four hundred and thirty (430) students sampled conveniently from the three year batches: 2012/13, 2013/14 and 2014/2015, of students admitted into the Department of Mathematics Education of the UEW, were given tests on the concepts of the arithmetic mean and the standard deviation.

This test was administered before formal lectures for the Introductory Probability and Statistics course began. The intention was to find out their conceptual understanding of the concepts of the arithmetic mean and the standard deviation based on their experiences and knowledge gained through the JHS core mathematics and the SHS core and elective mathematics curricula.

Nine of the participants were also randomly sampled and interviewed to gain more information to help the researcher describe their conceptual understanding and level of conceptual knowledge of the arithmetic mean and the standard deviation according to the APOS Theoretical Framework. The chapter also described in detail the different forms of data collection instruments and the procedures used in administering them.

CHAPTER FOUR

RESULTS/FINDINGS AND DISCUSSIONS

4.0 Overview

This chapter presents the results of the data analysis as well as a discussion of the major findings. The significant and novel findings identified are interpreted and discussed within the existing literature base. Data were collected and then analyzed in response to the problems posed in chapter one of this thesis. The purpose of this study was twofold. First, to investigate first year mathematics students' conceptual understanding of the "arithmetic mean" as a measure of central tendency. It also examined their conceptual understanding of the standard deviation as a measure of variation. The study further investigated the level of conceptual knowledge these students have on the concepts (i.e. arithmetic mean and the standard deviation) with respect to the Action, Process, Object and Schema Theoretical framework. In this chapter, the results are presented under the following themes in response to the research questions posed:

1. Demographic characteristics of respondents.
2. Research question one - First year undergraduate mathematics students' conceptual understanding of the arithmetic mean.
3. Research question two - Level of conceptual knowledge with respect to the APOS framework the undergraduate mathematics students have on the concept of the arithmetic mean.
4. Research question three - First year undergraduate mathematics students' conceptual understanding of the standard deviation.

5. Research question four- Level of conceptual knowledge with respect to the APOS framework the undergraduate mathematics students have on the concept of the standard deviation.

4.1 Demographic Characteristics of Respondents

Three consecutive year groups were used as the sample for the study and the data was collected over a period of three years. Table 4.1 shows the number of students used from each of the three-year groups.

Table 4. 1

Sample of the Year Groups

Year Group	N
2012/13	140
2013/14	140
2014/15	150
Total	**430**

As can be seen from Table 4.1, the sample for the study consisted of one hundred and forty (140) fresh undergraduate students from both the 2012/13 and 2013/14 academic year groups as well as one hundred and fifty (150) from the 2014/15 academic year group. The gender representation of these students from the various year groups is presented in Table 4.2.

Table 4. 2

Gender of Respondents

	Female		Male		All	
Year Group	**N**	**%**	**N**	**%**	**Total**	**%**
2012/13	17	12.0	123	88.0	140	32.6
2013/14	14	10.0	126	90.0	140	32.6
2014/15	15	10.0	135	90.0	150	34.8
Total	**46**	**10.7**	**384**	**89.3**	**430**	**100.0**

Out of the one hundred and forty participants for the 2012/13, 12.0% were females, whereas in both the 2013/14 and 2014/15 year groups, females formed 10.0% (Table 4.2). Females formed 10.7% of the total sample used for the study. The researcher looked at the age pattern of the respondents and this is displayed in Table 4.3.

Table 4. 3

Age Pattern of Respondents

	Age		Age		Age		
Year Group	**Up to 24**	**%**	**25 - 34**	**%**	**35 and Above**	**%**	**Total**
2012/13	122	87.1	16	11.4	2	1.5	140
2013/14	117	83.6	23	16.4	0	0.0	140
2014/15	124	82.7	24	16.0	2	1.3	150
Total	**363**	**84.5**	**63**	**14.6**	**4**	**0.9**	**430**

As indicated in Table 4.3, a majority, 87.1%, 83.6% and 82.7% of the 2012/13, 2013/14 and the 2014/15 year groups respectively were twenty-four (24) years or below. It was observed that 2.0% of both the 2012/13 as well as the 2014/15 year groups were thirty-five (35) years or above. For the 2013/14 year group, all participants were either thirty-four years (34) or below. Combining all groups, 84.5% of them were found to be twenty-four (24) years or below with only 0.9% of them to be thirty-five (35) years or above.

The researcher wanted to look at the representation of the regions in Ghana within which these students had their SHS education. This was to find out whether all the ten (10) regions in Ghana were represented in the sample. Table 4.4, therefore, shows the regional distribution for only the 2014/15 year group.

Table 4. 4

Regional Representation in 2014/15 Sample

Region	N	%
Greater Accra	7	4.7
Central	19	12.7
Eastern	15	10.0
Western	9	6.0
Ashanti	42	28.0
Brong Ahafo	25	16.7
Volta	17	11.3
Northern	4	2.7
Upper West	2	1.3
Upper East	10	6.6
Total	**150**	**100.0**

From Table 4.4, Ashanti region had the largest representation of 28.0%, followed by Brong Ahafo region with 16.7%. Representations of the Upper East, Eastern, Volta and Central regions ranged between 6.6% and 12.7%, and the least represented was Upper West region, with 1.3%.

The researcher was interested in the programs the respondents read at the SHS. This is because studies have shown that students who read the Sciences and the business programs have a more positive attitude towards mathematics than those in the arts (Karjanto, 2017), and this affects their performance in mathematics. The distribution of the students in the sample by programs is shown in Table 4.5.

Table 4. 5

Programs Studied by Respondents at SHS

Program	Sciences		Business		Arts		Total
Year Group	N	%	N	%	N	%	N
2012/13	61	43.6	41	29.3	38	27.1	140
2013/14	42	30.0	92	65.7	6	4.3	140
2014/15	87	58.0	61	40.7	2	1.3	150
Total	**190**	**43.9**	**194**	**45.2**	**46**	**10.9**	**430**

From Table 4.5 it was realized that for the 2012/13 year group, 43.6% studied science at the SHS with 29.3% studying business. For the 2013/14 and 2014/15 year groups, those who read either the sciences or business formed 95.7% and 98.7% respectively. Combining the year groups, only 10.9% of the participants studied the arts subjects. The sciences consisted of those who studied general science, agricultural science and the technical programs. It must however be noted that all the students admitted into the Department of Mathematics Education studied elective mathematics, which is a requirement for admission, at the SHS.

Again, from their responses, most of the students (82.7%) who formed the sample for the 2014/15 academic year completed SHS between 2012 and 2014. This meant that these respondents were either absorbed into the university straight from the SHS or had stayed in the house for a maximum of two years before being admitted. It was the expectation of the researcher that such a high percentage of students would retain a much of the things learnt at the SHS than the rest who had stayed in the house for a longer time before being admitted into the university.

It was noted that apart from 12 students (8.0%) from the 2014/15 batch who indicated they had studied mathematics above the SHS level, all the others, as well as those from the other year groups have studied mathematics up to the SHS level. The researcher was interested in this as it could help to determine the previous knowledge with which students used to write the test. The twelve (12) were found to be trained teachers who have taught mathematics either at the primary or the JHS level.

4.2 Research Question One - First Year Undergraduate Mathematics Students' Conceptual Understanding of the Arithmetic Mean.

Research question one sought to find out participants' conceptual understanding of the arithmetic mean. In this section, the students' overall performance of the test items and outcomes of the interviews used to examine their conceptual understanding of the arithmetic mean are presented. Refer to Appendix E for sample of student's responses on the test.

4.2.1 Students' Overall Performance on the Arithmetic Mean Test Items

In answering this question, students' were presented with a situation in Box 1 and were requested to explain the concept of the "arithmetic mean".

> a) A class of 160 sociology students took a final examination, and their arithmetic mean score was found to be 125.6. What does the "arithmetic mean" mean?
>
> b) What is the purpose of finding the arithmetic mean of a data set?

Box 1

As indicated in chapter three, the common responses (or themes) were identified and the percentages of students whose responses fell in a theme were noted. The most common responses were:

- sum over number of digits
- a typical value
- average
- representative value

Table 4.6 shows the analysis of the responses to the item in Box 1 by the three year groups.

Table 4. 6

Students' Responses to the Meaning of the Arithmetic Mean

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Sum Over Number of Digits	32	22.9	49	35.0	20	13.3	101	23.7
A Typical Value	0	0.0	4	2.9	0	0.0	4	1.0
Average	61	43.6	27	19.3	58	38.7	146	33.9
Representative Value	5	3.5	0	0.0	3	2.0	8	1.8
No Response	22	15.7	36	25.7	29	19.3	87	20.2
Others	20	14.3	24	17.1	40	26.7	84	19.4
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

It was discovered that, a majority of the 2012/13 and 2014/15 year groups (43.6% and 38.7%, respectively) described the arithmetic mean as "an average" whereas from the 2013/14 group, a majority (35.0%) described it using the computational algorithm: as "sum over number of digits". Only 3.5% of the 2012/13 batch, 2.9% of the 2013/14 batch and 2.0% of the 2014/15 batch could describe the arithmetic mean as either a "representative value" or a "typical value" of a data set. As many as between 30.0% and 46.0% across year groups either did not provide any response to the item or gave responses which had no bearing on the arithmetic mean.

Some of those responses included: "there was no deviation in the examination"; "out of the 160 students, 125.6 had the required score"; "some students did not take part in the examination"; "the class was below average"; "dividing the total frequency by the number of students who took the examination"; etc.

Bringing all the groups together, almost 34.0% of all the participants described the arithmetic mean as an average with 23.7% of them describing it with the computational algorithm "sum over number of digits". Less than 3.0% of all the respondents could describe the arithmetic mean as either a representative or a typical value. It is worth noting that close to 40.0% of them either did not respond to the item or gave wrong responses to the item.

The students were further asked to indicate the purpose for finding the arithmetic mean (Box 1). The students' responses on this item are presented in Table 4.7.

Table 4. 7

Students' Responses on the Purpose of Finding Arithmetic Mean

	Year Groups							
	2012/13		**2013/14**		**2014/15**		**All**	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
To get a representative Value	2	1.4	0	0.0	1	0.7	3	0.7
To get an the Average of Data	88	62.9	82	58.6	94	62.7	264	61.4
To get an the Midpoint of Data	4	2.9	3	2.1	0	0.0	7	1.7
No Response	29	20.7	41	29.3	25	16.6	95	22.2
Others	17	12.1	14	10.0	30	20.0	61	14.0
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

The responses in Table 4.7 show that between 58.6% and 62.9% of them across year groups referred back to the average. It also came out that 4 (2.9%) of the 2012/13 year groups and 3 (2.1%) of the 2013/14 year groups indicated that they find the arithmetic mean when we want the midpoint of a data set. Again, about a third of each year group (between 32.8% and 39.3%) either did not provide any response or gave others which included: "to know the collective performance of a class"; "to know the status of a data"; "to find the approximation per attempt"; "to determine the minimum mark each person would get"; etc.

The overall responses of the respondents indicate that a majority of them, 61.4%, have conceptualized that the arithmetic mean is found when the average of a data set is sought. About a third of them (36.2%) either gave wrong responses or could not provide any response to the item.

Participants were also given a discrete frequency distribution, and asked whether or not it could be possible for the arithmetic mean of the distribution to be 10, without doing any computation (see Box 2). This was to assess whether students are familiar with the property of the arithmetic mean which states that "the arithmetic mean of a data set can only take on values between the extremes" (Strauss & Bichler, 1988).

Below is the distribution of ages of a group of students in a class.

Ages of a Group of Students

Age	5	6	7	8	9
Frequency	2	3	3	2	1

A student said the arithmetic mean age of the group is 10 years.
Without doing any calculation, can this be true? Justify your answer.

Box 2

The responses of the participants for the three year groups when asked whether the arithmetic mean of a distribution can be 10, which is greater than the highest age in the distribution are represented in Table 4.8. The reasons for answering in the affirmative or not are shown in Tables 4.9 and 4.10 respectively.

Table 4. 8

Students' Responses to Whether the Arithmetic Mean can be 10

	Year Groups							
	2012/13		**2013/14**		**2014/15**		**All**	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Yes (Wrong)	24	17.1	20	14.3	12	8.0	56	13.1
No	96	68.6	82	58.6	125	83.3	303	70.2
No Response	20	14.3	38	27.1	13	8.7	71	16.7
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

Table 4. 9

Students' Justification of why the Arithmetic Mean Can be 10

	Year Groups							
	2012/13		**2013/14**		**2014/15**		**All**	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
From Calculation	8	33.3	6	30.0	2	16.7	16	26.7
The next number in the series is 10	2	8.3	1	5.0	0	0.0	3	4.4
No Response	14	58.4	13	65.0	10	83.3	37	68.9
Total	**24**	**100.0**	**20**	**100.0**	**12**	**100.0**	**56**	**100.0**

Table 4. 10

Student' Justification of why the Arithmetic Mean Cannot be 10

	Year Groups							
	2012/13		**2013/14**		**2014/15**		**All**	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
10 is greater than the Highest Value	1	1.0	0	0.0	9	7.2	10	2.7
Data has Small Range	0	0.0	0	0.0	2	1.6	2	0.5
From Calculation	52	54.2	47	57.3	56	44.8	155	52.1
10 is not in the middle of the data	2	2.1	0	0.0	0	0.0	2	0.7
No Response	35	36.4	25	30.5	22	17.6	82	28.2
Others	6	6.3	10	12.2	36	28.8	52	15.8
Total	**96**	**100.0**	**82**	**100.0**	**125**	**100.0**	**303**	**100**

The responses show that between 8.0% and 17.1% across year batches responded in the affirmative, whiles a majority of all the year groups, between 58.6% and 83.3% responded in the negative (Table 4.8). The responses provided by between 16.7% and 33.3% for all year groups, who answered in the affirmative, indicated that their justification can be attributed to the outcome they obtained after computing the arithmetic mean of the data. Nevertheless, a majority from each year group (between 58.4% and 83.3%) as can be seen in Table 4.9 could not justify their responses.

Though the participants were not to do any computation, from Table 4.10, between 44.8% and 57.3% of those who responded in the negative also attributed their response to the answer they had after computing the arithmetic mean for the data. Only 1.0% of the 2012/13 batch and 7.2% of the 2014/15 batch could point out that "10 was greater than the highest value of the distribution" hence it could not be the arithmetic mean of the data set. The rest either could not give any justification for their answer or gave other reasons, some of which are: "in calculating ages there is no decimal value", "this can be answered only after computation, one cannot look at the question and provide an answer", "arithmetic mean is the summation of frequencies and it is not 10", "the arithmetic mean is not like the mode which can be read from a table without computing", etc.

With the overall responses of the item in Box 2, it can be concluded that though three hundred and three (303), representing 70.2%, of the respondents correctly indicated that the arithmetic mean cannot be 10 (Table 4.8), only 12 (representing 3.4%) of them gave a correct justification that either "10 is greater than the highest age in the data" or "10 is not in the middle of the data". Though a majority of the respondents had defined the arithmetic mean as an average, they could not use that to justify their affirmative response since they did not understand what an average is. They could only use their computational understanding as a justification.

The participants were also asked to compute the arithmetic mean of the data in the frequency table in Box 2. Table 4.11 shows the distribution of students who were able or unable to compute the arithmetic mean correctly.

Table 4. 11

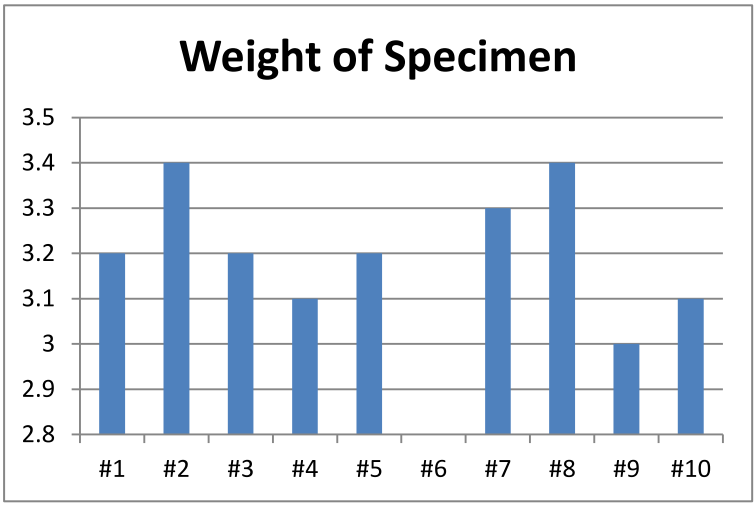
Distribution of Students Computing the Arithmetic Mean of Data Presented in Frequency Table

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Correct	107	76.4	95	67.9	101	67.3	303	70.5
Wrong	28	20.0	41	29.3	43	28.7	112	26.0
No Response	5	3.6	4	2.8	6	4.0	15	3.5
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

It is indicated from Table 4.11 that between 67.9% and 76.4% of participants across year groups were able to compute it correctly. The non–response rate for this item was reduced drastically to 4.0% and below for all year groups. This implies that if the students have data presented in a frequency table majority of them can compute the arithmetic mean. As a combined group, 70.5% of all the respondents were able to compute the arithmetic mean of the frequency table in Box 2.

Again, participants were also asked to use the arithmetic mean of data presented in a bar chart to determine the value of a missing bar (see Box 3). This item was adapted from Marnich's (2008) pilot study. Table 4.12 shows the distribution of students who were able or unable to determine the missing number using the given arithmetic mean.

In a chemistry lab a student weighed a specimen ten times. The results of those weighings are presented in Figure 4.1 below. The student lost the 6th weighing of the specimen after she calculated the arithmetic mean of the ten weighings to be 3.2 as indicated by the dark line in the graph below. What could have been the value for the 6th weighing if the arithmetic mean is 3.2?



Box 3

Table 4. 12

Distribution of Students who were able or unable to determine the missing number using the given arithmetic mean

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Correct	1	0.7	2	1.4	4	2.7	7	1.6
Wrong	58	41.4	49	35.0	77	51.3	184	42.6
No Response	81	57.9	89	63.6	69	46.0	239	55.8
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

Table 4.12 indicates that less than 3.0% of participants from all the year groups were able to answer this item correctly. Majority from each of the year groups (between 46.0% and 63.6%) did not attempt to give an answer to the test item. The rest who attempted it had it wrong. As a common group, only 1.6% of all the

respondents from all year groups were able to find the missing number from the bar chart. More than half (55.8%) did not attempt this question due to the lack of confidence to solve it. This reveals that though students can confidently compute the arithmetic mean of a data set, reversing the process is a challenge.

Finally, on the arithmetic mean, two data sets were presented to participants and asked to determine which of them would have a larger arithmetic mean (see Box 4). This item was adapted from the ARTIST website. Table 4.13 shows the distribution of students who were able or unable to determine which of them would have a larger arithmetic mean. Tables 4.14 and 4.15 present students' explanations for selecting data set A or B to have the larger arithmetic mean respectively.

Kwesi and Adwoa are analyzing the following data sets.
Data Set A: 110, 112, 114, 115, 116, 118.
Data Set B: 2, 6, 15, 28, 59, 112.
Without calculating, which data set, A or B, would have the larger arithmetic mean? Explain your answer.

Box 4

Table 4. 13

Distribution of Students who were able or unable to determine which of Two Data sets has a Larger Arithmetic Mean

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	N	%	N	%	N	%	N	%
Data Set A (Correct)	110	78.6	120	85.7	131	87.3	361	83.9
Data Set B (Wrong)	12	8.6	9	6.4	10	6.7	31	7.2
No Response	18	12.8	11	7.9	9	6.0	38	8.9
Total	140	100.0	140	100.0	150	100.0	430	100.0

Table 4. 14

Students' Explanation to why Data Set A has the Larger Arithmetic Mean

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	N	%	N	%	N	%	N	%
Larger Numbers	79	71.8	83	69.2	91	69.5	253	70.2
From Calculation	10	9.1	16	13.3	3	2.3	29	8.2
Larger Sum	3	2.7	0	0.0	0	0.0	3	0.9
Small Range	0	0.0	4	3.3	1	0.8	5	1.4
No Response	11	10.0	12	10.0	7	5.3	30	8.4
Others	7	6.4	5	4.2	29	22.1	41	10.9
Total	**110**	**100.0**	**120**	**100.0**	**131**	**100.0**	**361**	**100.0**

Table 4. 15

Students' Explanation to why Data Set B will have a Larger Arithmetic Mean

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	N	%	N	%	N	%	N	%
The numbers are close together	0	0.0	0	0.0	2	20.0	2	6.7
From Calculation	5	41.7	4	44.4	1	10.0	10	32.0
Large Range	2	16.6	0	0.0	0	0.0	2	5.5
No Response	5	41.7	5	55.6	7	70.0	17	55.8
Total	**12**	**100.0**	**9**	**100.0**	**10**	**100.0**	**31**	**100.0**

Responses from Table 4.13 show as many as between 78.6% and 87.3% across all year groups correctly selecting Data Set A as the one with the larger arithmetic mean. The non-response rate for this item was 12.8%, 7.9% and 6.0% from the 2012/13, 2013/14 and 2014/15 batches respectively. Explaining their choice of Data Set A as the one with the larger arithmetic mean as revealed in Table 4.14, it can be seen that around 70.0% of all year groups appropriately pointed out that Data Set A has larger numbers than those of Data Set B. Only a few (between 2.3% and 13.3%) across all year groups based their argument on computations even though they were not to do so. The non-response rate for this item was also smaller compared to when participants had to explain or define the arithmetic mean (8.4% for the combined group). Some of the other reasons given included: “Data set A has many values than Data set B”, “set A has high values of frequency”, “set A has more set of numbers”, etc.

From Table 4.15, it is revealed that more than half of those who wrongly selected Data Set B could not explain their choice of answer; however a significant number of them attributed it to the results obtained after computing the arithmetic mean of the two Data Sets. This implies those ten (10) individuals out of the thirty one (31), representing 32.0% could not correctly compute the arithmetic mean of the two Data Sets

4.2.2 Interview Results on Students’ Conceptual Understanding of the Arithmetic Mean

In addition to the responses to the test items reported above, nine of the students were randomly selected and interviewed to find out their conceptual understanding of the arithmetic mean. The responses from the participants are presented in the paragraphs that follow:

Interviewer: What do you understand by the term “arithmetic mean" in statistics?

Ato: Um,..., it is the sum of all values in a set, divided by the number of values that were added.

Interviewer: Good, but this sounds more of how it is calculated right?

Ato: Yes.... (Smiles).

Interviewer: So what does the arithmetic mean tell us about the data set from which it was calculated?

Ato: It represents the average of the set.

Interviewer: Where average means?

Ato: Uh? ……. .. (Looks surprised), ….. It is the number obtained when you add and divide the numbers in a set. You will always get a single number.

Interviewer: Ok. So what does that single number measure or what does it represent?

Ato: I said it represents the average.

Interviewer: Are there any other averages you know of?

Ato: No! There is only one average, the arithmetic mean, and that is how we find it.

Interviewer:As a teacher, if you calculate the arithmetic mean scores of two of your classes taking the same course, of what use will those values be for you?

Ato: I think you will know the people who did well and those who didn't do well,… right? (He asks the interviewer)….. But I don't know how, I was only taught how to calculate the arithmetic mean so I am not sure how.

Interviewer: So given a set of numbers like, {2, 5, 11, 7, 10}, what will be the arithmetic mean?

Ato: (He does his computations). The arithmetic mean is 7.

Interviewer: Good! Can this number, 7, be a true representation of the numbers in the set?

Ato: No! (Looks a little serious and confused). That will not be possible since we have different numbers. It is just the average of the numbers.

Interviewer: Any other thing you know about the arithmetic mean? Is there anything to be added to what you have said already?

Ato: No Madam

Interviewer: Ok! Thank you (shakes his hand). We shall continue the discussion in class.

Ato: Thank you too, Madam (smiles).

That was how far interviewee Ato could go. The same views were presented by six other interviewees: Esinam, Mohammed, Laryea, Korshi, Ebo and Dakpo. They swung between the "add them up and divide algorithm" and the fact that the arithmetic mean is an average. However, they could not explain what an average is, let alone mention other examples of averages. To them, the arithmetic mean is an average, the only average, whilst the average is the outcome of the "add them up and divide algorithm".

Though they all exhibited mastery over the computational algorithm, when further asked what the arithmetic mean represents, they just went back to the computational algorithm and the average, without knowing what the average represents. Though Kwesi was able to mention the other averages and how they are found, he was also not able to tell what the averages represents as can be seen in the extract below:

Interviewer: What is meant by the term "arithmetic mean" in statistics?
Kwesi: Should I define it or tell how it is calculated? (Scratches the head and makes a face as well).
Interviewer: Tell me all you know about it.
Kwesi: I am not sure, but I know it is found by using this formula. (Writes the "add them up and divide algorithm" down). ... You add all the numbers and divide by the number of them.
Interviewer: Then, what will your answer represent?
Kwesi: That is the arithmetic mean. Sometimes if you are not asked to find the mean they will ask you to calculate the other one..... (Tries to remember),... the median, yes, median.
Interviewer: Which one is that?
Kwesi: That is also like the arithmetic mean, but how to find it is different?
Interviewer: Ohh, so how are the two related?
Kwesi: Yes. I know they are both averages ,... and the mode too. For the mode you can just look at the histogram and mention it, whereas the median... (Umm) you have to arrange them and select from the middle.
Interviewer:So collectively, what do they represent?
Kwesi: That one I don't know. I wasn't taught.
Interviewer: When do you normally find them?
Kwesi: It is only when you are asked to.

In the same vein, Tei also used the computational algorithm to define the mean. He further described it as an average and mentioned what it represents as can be seen in the extract below:

Interviewer: What do you understand by the term "arithmetic mean" in statistics?

Tei: The arithmetic mean is the sum of all numbers in a set divided by the number of the values that were added.

Interviewer: Any other thing you know about the arithmetic mean?

Tei: It is an average. ..... In fact, one of the three averages that can be calculated from a given set of numbers.

Interviewer: What then are these averages and what information do they give?

Tei: (Umm, ……. (Looks up), I think the middle or center of the numbers. Yes. So we call them measures of center… or measures of central tendency. That is why they measure the center; however, we normally calculate the arithmetic mean than the rest.

Interviewer: Good. So for a set of data like: {55, 60, 69, 80, 98, 105 and 120}, can the arithmetic mean be 125? Without doing any calculation.

Tei: I will have to do my own calculation first before any comment.

Interviewer: Is the 125 reasonable, considering your view of an average being the center of a data set?

Tei: (Makes a face and pauses, ... (Tries to think)……. I can only comment when I have done my own calculation.

Interviewer: What else do you know about the arithmetic mean?

Tei: That is all I know. We were only taught to calculate at SHS, so that is all I know.

Interviewer: Thank you. We shall continue the discussions in class.

Tei: Thank you too.

The responses from the interview have indicated that a majority of the participants across the year groups have conceptualized the arithmetic mean as an average and also a computational act with the add them up and divide algorithm.

4.2.3 Discussion of Results on Students' Conceptual Understanding of the Arithmetic Mean

The arithmetic mean has been conceptualized in different ways by different researchers to help students develop a better conception of it. It has been conceptualized by the fair share and balance models (Pollatsek, Lima, & Well, 1981; Hardiman, Well, & Pollatsek, 1984), as well as being characterized as a "typical" or a "representative value" of a data set (Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988). Thompson (1998) also looked at the multiplicative conception of the arithmetic mean, where it measures group performance in relation to the number of contributors in the group.

Responses from both the test and the interview of the current study have indicated that a majority of the participants across the year groups have conceptualized the arithmetic mean as an average and also a computational act with the "add them up and divide algorithm". From Table 4.6, only a handful of the participants (less than 4.0% across the year groups) in the test, described the arithmetic mean as either a typical or a representative value of a data set as described by some researchers.

Though participants from the present study described the arithmetic mean as an average, not even one of them described or explained what averages represented in the test. Neither did they make mention of the other averages to help distinguish the arithmetic mean from the other averages. Only two participants out of the nine interviewed, mentioned the other averages. Kwesi mentioned the median and mode and how they are found without demonstrating understanding of what these averages represent. Only Tei went further to indicate that the averages represent the center of a data set and that they are mostly called measures of central tendency.

Even though they also defined the arithmetic mean as a computational act, with the "add them up and divide algorithm", and also demonstrated competence in the computation of the arithmetic mean of a distribution, they did not demonstrate a conceptual understanding of the outcome of this act (Mokros & Russell, 1995; Cai, 1998; McGatha, Cobb, & McClain, 2002; Groth & Bergner, 2006). This is what Pollatsek, Lima, and Well (1981) referred to as students' "knowledge of the arithmetic mean seems to begin and end with an impoverished computational formula" (p. 191). Though using the computational algorithm in a way tries to distinguish the arithmetic mean from how the other averages are obtained, it does not necessarily tell what it is. It must also be noted that knowing how to compute the arithmetic mean does not necessarily guarantee its conceptual understanding.

Similar to the above results is the pretest result of Dubreil-Frémont, Chevallier-Gaté, and Zendrera (2014). They observed in their study that among the 352 students, 61.2% also saw the arithmetic mean as a computational act and as such used the "add them up and divide" algorithm to define it. Sixty seven (19.0%) could not give any response with 14.2%, giving wrong definitions for the arithmetic mean.

Participants' responses in the current study, as revealed in Table 4.6 indicated that between 30.0% and 46.0% across year groups either could not describe or explain what the arithmetic mean is or gave inappropriate answers. Such a high percentage raises an eyebrow on student's inability to explain what the arithmetic mean is. Again, when participants were asked about the purpose for finding the arithmetic mean, between 58.6% and 62.9% (Table 4.7) of each year group pointed out that it is to find the average of a data set. The arithmetic mean, according to many of the participants, is an average and it is computed when we want the average of a data set, by adding the data values and dividing by the number of values. They just go round the term "average" to describe everything about the arithmetic mean without an understanding of what an average is.

It was observed that the high percentage of no-response (between 15.7% and 25.7% from Table 4.6) on the item requesting participants to explain what the arithmetic mean is, was drastically reduced (to between 2.8% to 4.0% from Table 4.6) when they had to compute the arithmetic mean. This implied that participants were more confident and able in computing the arithmetic mean than in explaining what it represented.

Various researchers have acknowledged that the concept of the arithmetic mean is rather a difficult concept to understand by both children and adults. Strauss and Bichler (1988) observed fourth to eighth graders to have difficulty in understanding the properties of the mean. In the work of Pollatsek, Lima, and Well (1981), they found that most college students, though might be able to use the arithmetic mean algorithm, could not determine whether the algorithm might or might not present the correct response. The students did not also show an understanding of what their results represented.

Similarly, in their study of the properties of the arithmetic mean, Leon and Zawojewski (1990) found out that it was easier for most people to understand the arithmetic mean as a computational construct, but difficult to understand it as a representative value. Among children aged 13 to 14 years, Goodchild (1988) found that students have difficulty with the representativeness and expectation interpretations of the arithmetic mean. Exploring students' conceptual understanding of the averaging algorithm, Cai (1995) also found that a majority of sixth–graders, though, knew the computational algorithm for the arithmetic mean, only a few had a conceptual understanding of the concept just as observed in the current study.

Bremigan (2003) have indicated that among the numerical statistical measures presented in introductory statistics courses, though the concept of the arithmetic mean appears to be the first to be presented, it is done in the early grades where the focus is on application of division instead of it as a statistical measure. As a result, Bremigan (2003) pointed out that, the algorithm of finding the arithmetic mean of a set of data, though might be known and seen as relatively easy to students, requiring only the addition and division skills, the relationship between the outcome of this algorithm and the data set from which it was computed might be unknown.

From participant's responses from both the test and the interview, it can be concluded that after going through the JHS and SHS syllabi, first year undergraduate mathematics students have conceptualized the arithmetic mean to be an average as well as a computational act with the "add them up and divide algorithm". However, though they define the arithmetic mean as an average, they cannot explain what averages are or what they represent. When asked what averages represent, they go back to the "add them up and divide algorithm". They also do not consider the median and mode as averages.

As a computational act, they have conceptualized the arithmetic mean as the "sum of numbers in a data set, divided by the number of values in the data set". Though they are all proficient in this act or skill, they do not understand what the outcome of this act or skill represents or its relation with the set from which it was computed from. They just move between the "arithmetic mean is an average" and "it is the sum of data values divided by the number of data values".

4.3 Research Question Two - Level of Conceptual Knowledge with Respect to the APOS Framework the Undergraduate Mathematics Students have on the Concept of Arithmetic Mean

From participant's responses to the test items, it was revealed that a majority (70.5% of overall participants) was able to compute the arithmetic mean of the discrete frequency table given without being provided with any external cues. During the interview too, all interviewees showed mastery in the computation as well as in describing the algorithm for finding the arithmetic mean. This indicates that participants are not limited to an action conception of the arithmetic mean, which is the least level according to the APOS Theoretical Framework. In the APOS theory, if a student is limited to an action conception of the arithmetic mean, such a student will not be able to compute the arithmetic mean in the absence of a given formula. However, participants in the current study demonstrated that the action of averaging is fairly simple and easy since it has been done many times at the JHS and SHS levels.

In the APOS theory, when an individual performs an action many times, it may be interiorized into a process. This means the individual can describe the action whilst reflecting on it, without actually performing it and without any external prompts. By this, the individual is said to have built internal constructions that perform the action without any external incitements. Therefore, an individual with a process conception of a mathematical concept can describe the process and even reverse it whiles reflecting on it, without performing the action itself. Whereas an action is considered to be an external act under the APOS Framework, that is, what an individual does under external cues, process is internal and under the individual's own control.

In the present study, as revealed in Table 4.11, on participants' responses to the computation of the arithmetic mean from a frequency table, between 67.3% and 76.4% (averaging 70.5% for the combined group) were able to compute the arithmetic mean correctly. Most of those who had it wrong made mistakes during the computation process. However, they did not demonstrate an understanding of the computed arithmetic mean in relation to the data from which it was computed from. To them, it just represented "an average with no meaning". Aside the computational act, participants did not demonstrate an understanding of any property of the arithmetic mean.

To move to a process conception of a concept, an individual must be able to interiorize the actions on the concept, be able to reflect on procedures applied to a particular process and become aware of the process as a totality. He must also be able to realize that the entity has properties on which transformations can act. These transformations may be actions or processes, and he must be able to actually construct such transformations. When this happens, we say that the individual has encapsulated the process into an object, and that he has an object conception of the concept (Asiala et al., 1997). According to Piaget, encapsulating a process is a mental object in which physical or mental actions are "reconstructed and reorganized on a higher plane of thought" to become understood by an individual (Beth & Piaget 1966, p. 247).

In the current study, though participants demonstrated an action conception of the arithmetic mean, they did not demonstrate an understanding of the arithmetic mean as an entity with meaningful properties. They could not tell what the outcome of their action, the computed arithmetic mean, represented. In the test, less than 3.0% of the combined participants from all year groups mentioned that the arithmetic mean is either a representative or a typical value of a data set and that we find them when we want the midpoint of the data set (Tables 4.6 and 4.7). To them the arithmetic mean is an average without knowing what an average is, let alone give examples.

Throughout the test, participants were silent on the other averages and their relationship with the arithmetic mean, except in the interview when upon probing Kwesi mentioned the median and mode as other averages and described how they are found. Nevertheless, he could not tell what they represent. Only Tei exhibited some sort of understanding of what the averages are, by indicating they represent the center of a data set and that they are called measures of center. Even so, he was deficient in identifying that a value outside the range of values in a data set cannot be the arithmetic mean of the data. He insisted on doing his own computation before giving an answer.

Just as Tei exhibited during the interview, during the test participants did not demonstrate the knowledge of the property that “the arithmetic mean of a data set can only take on values between the extremes”. From Table 4.8, though a majority (70.2% of the combined group) pointed out that the arithmetic mean of the given data set could not be 10, i.e. higher than the largest value in the data set, it is evident from Table 4.10 that they could not justify their answer without doing any calculations. Only 3.4% of the combined group gave appropriate justifications as either “10 is larger than the highest age in the data” or “10 is not in the middle of the data set”. The non-response rate or those who gave wrong justifications were high (44.0%); implying participants’ confidence in justifying their answer was low.

In interiorizing actions to process, the individual must be able to reverse the steps of the transformations that act on them. Nevertheless, in the current study, only 1.6.0% of the overall group was able to reverse the process of the computation of the arithmetic mean from the bar chart provided (Table 4.12). By this description, it can be concluded that participants exhibited an incomplete process conception of the arithmetic mean.

It must be noted that though the APOS framework has been used in other studies like APOS-Sampling (Hatfield, 2013); APOS–Limits (Maharaj, 2010), APOS–Algebra (Arnawa, Sumarno, Kartasasmita, & Baskoro, 2007), they all concentrated on using the mental processes of the framework to help students construct mathematical knowledge of the concepts involved. Only those of Mathews and Clark (2007) and Clark, Kraut, Mathews, and Wimbish (2007) are known to the researcher to have used it to analyze students' level of conceptual knowledge of some statistical concepts.

Thus, results of the current study are consistent with that of Mathews and Clark's (2007) study. In their study, it was observed that even successful undergraduate students who had completed an elementary statistics course with a grade of "A" still acknowledged the arithmetic mean as a process of computation. They had moved past an action conception of the arithmetic mean to at least a process conception. The computational algorithm was clearly seen to be under their control. Students, in Mathews and Clark's (2007) study, did not see the arithmetic mean as a measure of central tendency, an object possessing property, nor as a cognitive entity. According to the students, the arithmetic mean is only a process of computation.

Also consistent with the results of the current study are that observed by, Clark, Kraut, Mathews, and Wimbish (2007). Extending the participants in Mathews and Clark's (2007) study, they also observed that all participants were not limited to the action conception of the arithmetic mean. As the action of computing the arithmetic mean was seen to be relatively simple. However, though students were able to compute the arithmetic mean, describe its computational algorithm, and reverse this process in some cases, these students did not see the arithmetic mean of a set of data as an entity in itself. They did not read any meaning into the computed arithmetic means.

Three of the participants in the Clark, Kraut, Mathews, and Wimbish's (2007) study were found not to have moved beyond a process conception of the arithmetic mean. They were not able to associate any meaning to the means they calculated. The students were able to compare the arithmetic mean with the median, which is also a measure of location. They demonstrated the knowledge that the arithmetic mean is a number that can be used to represent a whole set of data. It was nevertheless found that there was still an over reliance on the term average and the computational algorithm.

Research question two also sought to investigate the level of conceptual knowledge of the arithmetic mean that first year undergraduate mathematics students reach with respect to the APOS Theory. From the results and discussions from both the test and the interview, it was concluded that though first year undergraduate mathematics students have moved past an action conception of the arithmetic mean, they have an incomplete process conception of it according to the APOS Theory. Though they can describe the algorithm for the computation of the arithmetic mean without any external prompts, they could not demonstrate a conceptual understanding of the outcome of the algorithm. Also, they did not see the arithmetic mean as an entity with any property; neither could they reverse the process of the algorithm.

4.4 Research Question Three - First Year Undergraduate Mathematics Students' Conceptual Understanding of the Standard Deviation as a Measure of Variation.

Research question three also sought to examine participants' conceptual understanding of the standard deviation. This section, presents students' general performance on the test items and interview results of the standard deviation.

4.4.1 Students' Overall Performance on the Items on the Standard Deviation.

The main item on the standard deviation was to find from participants how they understand the concept of the standard deviation. The item is presented in Box 5: Table 4.16 shows participants' explanation of the concept of the standard deviation.

> Here are values of measurements of distances (in millimeters) between the two eyes of 8 adult patients who were to be given eyeglasses: 67, 66, 59, 62, 63, 66, 55, and 66. The mean distance is 63 mm and the standard deviation is 4.2. Explain to someone not in a statistics class, what the standard deviation means.

Box 5

Table 4. 16

Students' Explanation of the Concept of Standard Deviation

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Deviation from the mean	1	0.7	1	0.7	2	1.3	4	0.9
Spread of Data	1	0.7	2	1.4	6	4.0	9	2.0
Using Calculation	4	2.9	3	2.1	11	7.3	18	4.1
Range	1	0.7	0	0.0	0	0.0	1	0.2
Numbers that Vary	4	2.9	5	3.6	0	0.0	9	2.2
No Response	105	75.0	89	63.6	78	52.0	272	63.6
Others	24	17.1	40	28.6	53	35.4	117	27.0
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

It is revealed in Table 4.16 that only 1.4%, 2.1% and 5.3% of the 2012/13, 2013/14, 2014/15 batches respectively, could describe the standard deviation as either "a deviation from the arithmetic mean" or "the spread out of the data". The non-response rate for this item (between 52.0% and 78.0% for all year groups) was too high, likewise those who provided inappropriate responses (between 17.1% and 35.4% for the year groups). Most of those who tried to use the algorithm to explain what the standard deviation is, even quoted it wrongly. Some of the inappropriate responses included: "it means those who passed the eyeglasses test"; "standard deviation represents measurements that are close together"; "it means the measurement ranges, but at a higher pace"; "it means the original measurements were in the range of 4.2"; "it means the mean of the measurements was adjusted by 4.2"; etc.

Participants were also asked to compute the standard deviation of the frequency distribution in Box 2. The distribution of those who were able and unable to compute the standard deviation is presented in Table 4.17.

Table 4. 17

Distribution of those who were able and unable to compute the Standard Deviation

	Year Groups							
	2012/13		**2013/14**		**2014/15**		**All**	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Correct	12	8.6	15	10.7	8	5.3	35	8.2
Wrong	60	42.8	53	37.9	77	51.3	190	44.0
No Response	68	48.6	72	51.4	65	43.4	205	47.8
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

As reflected in Table 4.17, computation of the standard deviation was a real challenge to participants. Only between 5.3% and 10.7% of the participants across year groups were able to compute it correctly. The rest (between 89.0% and 94.0% across year batches) either did not respond to the item or computed it wrongly.

To further investigate participants' understanding of the concept of the standard deviation, they were presented with the two Data Sets: A and B, to indicate which of them will have a larger standard deviation and justify their responses. The item as given is presented in Box 6. Participants' responses to the data set that will have the larger standard deviation are displayed in Table 4.18 and their justification to the choice of data set B or A, presented in Tables 4.19 and 4.20 respectively.

> Kwesi and Adwoa are analyzing the following two data sets.
> Data Set A: 110, 112, 114, 115, 116, 118.
> Data Set B: 2, 6, 15, 28, 59, 112.
> a) Without calculating, which of the data sets, A or B, will have the larger standard deviation?
> b) Justify your answer to (a).

Box 6

Table 4. 18

Students' Responses on the Data Set that will have the Larger Standard Deviation

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Correct (Data Set B)	74	52.8	85	60.7	77	51.3	236	55.0
Wrong (Data Set A)	54	38.6	49	35.0	40	26.7	143	33.4
No Response	12	8.6	6	4.3	33	22.0	51	11.6
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

Table 4. 19

Students' Justification for Data Set B to have a Larger Standard Deviation

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
More Spread Out	10	13.5	6	7.1	12	15.6	28	12.1
Higher Range	0	0.0	3	3.5	0	0.0	3	1.1
Smaller Numbers	5	6.8	13	15.3	9	11.7	27	11.3
Lower Mean	15	20.3	19	22.4	23	29.9	57	24.2
High Frequency	3	4.0	2	2.3	0	0.0	5	2.1
From Calculation	10	13.5	6	7.0	1	1.3	17	7.2
No Response	16	21.6	26	30.6	11	14.3	53	22.2
Others	15	20.3	10	11.8	21	27.2	46	19.8
Total	**74**	**100.0**	**85**	**100.0**	**77**	**100.0**	**236**	**100.0**

Table 4. 20

Students' Justification for Data Set A to have a Larger Standard Deviation

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Large Numbers	10	18.5	5	10.2	5	12.5	20	13.7
From Calculation	4	7.4	6	12.2	8	20.0	18	13.2
Close together	0	0.0	4	8.2	0	0.0	4	2.7
No Response	27	50.0	22	44.9	19	47.5	68	47.5
Others	13	24.1	12	24.5	8	20.0	33	22.9
Total	**54**	**100.0**	**49**	**100.0**	**40**	**100.0**	**143**	**100.0**

It can be seen from Table 4.18 that between 51.3% and 60.7% correctly selected Data Set B to have a larger standard deviation. The non-response rate for this item also reduced, especially for the 2012/13 and 2013/14 year groups indicating participants' high confidence in answering the item. However, when they had to justify their response for correctly selecting Data Set B, as shown in Table 4.19, only between 10.6% and 15.6% across year groups identified Data set B as either having a wider spread or a higher range than Data Set A. Among the other responses, between 20.3% and 29.9% across year batches showed that the larger standard deviation of Data Set B is as a result of its lower mean. Some of the other reasons provided included: "standard deviation is inversely proportional to the sum of a data set"; "set B has a larger frequency"; "standard deviation is not affected by the arithmetic mean"; etc.

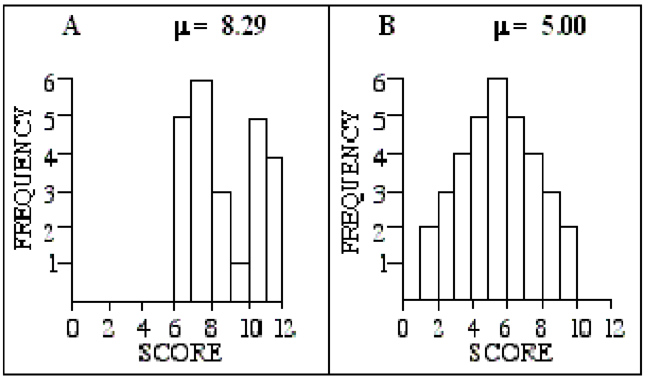
For the justification of those who erroneously selected Data Set A as having the larger standard deviation, as indicated in Table 4.20, nearly half (between 47.5% and 50.0%) of the participants across year groups could not give any justification for their answer. Some attributed their response to the results they had when they computed the standard deviation of the data even though they were not to do any computation. Other reasons given included: "Data Set A has larger numbers"; "It has a larger mean"; "the standard deviation is dependent on x so the higher the x the higher the standard deviation"; "the data values in set A are more than that in B"; etc.

Similar to the item in Box 6, participants were to indicate among a pair of Histogram, (A and B), which one will have a larger standard deviation and explain their answer. This is presented in Box 7. Table 4.21 displays participant's ability or inability to correctly select the Histogram with a larger standard deviation, with the explanation of their wrong or correct responses presented in Tables 4.22 and 4.23 respectively.

For the pair of histogram A and B:

a) determine which one has the larger standard deviation (it is not necessary to do any calculations to answer this question).

b) explain your answer.



Box 7

Table 4. 21

Participant's Ability or Inability to select the Histogram with a Larger Standard Deviation

	Year Groups							
	2012/13		**2013/14**		**2014/15**		**All**	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Histogram A (Wrong)	57	40.7	55	39.3	78	52.0	190	44.0
Histogram B (Correct)	40	28.6	46	32.9	47	31.3	133	30.9
Same Standard Deviations	8	5.7	10	7.1	5	3.3	23	5.4
No Response	35	25.0	29	20.7	20	13.4	84	19.7
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

Table 4. 22

Students' Explanations to "why Histogram A has a Larger Standard Deviation

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Less Spread Out	3	5.3	1	1.8	2	2.6	6	3.2
Higher Mean	9	15.8	7	12.7	14	17.9	30	15.5
Widely Spread	3	5.3	1	1.8	0	0.0	4	2.4
Performance was Poor	2	3.5	4	7.3	3	3.9	9	4.9
No Response	23	40.3	25	45.5	18	23.1	66	36.3
Others	17	29.8	17	30.9	41	52.5	75	37.7
Total	**57**	**100.0**	**55**	**100.0**	**78**	**100.0**	**190**	**100.0**

Table 4. 23

Students' Explanations to "why Histogram B has a Larger Standard Deviation

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
More Spread Out	4	10.0	2	4.4	1	2.1	7	5.5
Lower Mean	10	25.0	11	23.9	8	17.0	29	22.0
Distribution is More Uniform	2	5.0	0	0.0	0	0.0	2	1.7
More people Below Average	0	0.0	3	6.5	0	0.0	3	2.2
No Response	17	42.5	22	47.8	16	34.1	55	41.4
Others	7	17.5	8	17.4	22	46.8	37	27.2
Total	**40**	**100.0**	**46**	**100.0**	**47**	**100.0**	**133**	**100.0**

It is revealed in Table 4.21 that between 39.3% and 52.0% wrongly selected Histogram A to have a larger standard deviation while between 28.6% and 32.9% of all year groups correctly selected Histogram B. Giving explanations for wrongly selecting Histogram A to have a larger standard deviation as depicted in Table 4.22, between 12.7% and 17.9% attributed the larger standard deviation of Histogram A to its larger arithmetic mean. Though other reasons were given as reflected in Table 4.22, the majority could not justify their answers (between 23.1% and 45.6% across year groups). Other reasons given that were not tabulated included: “arrangement of bars in Histogram B is better than in Histogram A”; “Histogram A has the highest bar”; “Histogram A has a few scores”; “Histogram A has the higher frequencies than Histogram B” (who was actually referring to many bars and as such many data values in A than in B); “Histogram A has a greater difference in its marks than Histogram B”; “entries for height of Histogram A does not follow the correct order”; etc.

Though a majority of the participants correctly selected Histogram B to have a larger standard deviation, only 10.0% of the 2012/13 batch, 4.4% of the 2013/14 batch and 2.1% of the 2014/15 (see Table 4.23) batch could correctly explain that Histogram B is more spread out than Histogram A. Though other reasons were given, as shown in Table 4.23, it could be seen that between 17.0% and 25.0% of all year batches accredited the larger standard deviation of Histogram B to its lower mean. Quite a number also failed to justify their answer (between 34.1% and 47.8% across year groups). Some of the other reasons given included: “Histogram B has more frequencies”; “Histogram B has more scores than Histogram A”; “In Histogram B, all the scores have their respective frequencies”; “Histogram B has closer intervals than Histogram A”; “the set of data for Histogram B is more than that of Histogram A”; “the answer gain for Histogram B is smaller than that of Histogram A”; etc.

The last test item on the standard deviation requested that participants explain what it means for the standard deviation of a set of examination scores to be zero. The situational item is presented in Box 8 and students' responses illustrated in Table 4.24.

Imagine a set of quiz scores. If you were told that the standard deviation of the quiz scores is 0, what would that tell you about the scores?

Box 8

Table 4. 24

Student's Responses on what a Standard Deviation of Zero means

	Year Groups							
	2012/13		2013/14		2014/15		All	
Response	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
All Values are the same	10	7.1	21	15.0	17	11.3	48	11.1
Wrong Calculation	9	6.4	7	5.0	2	1.3	18	4.2
Poor Performance	22	15.7	18	12.8	16	10.7	56	13.1
All values are zero	12	8.6	3	2.1	9	6.0	24	5.6
There were no scores	0	0.0	4	2.9	0	0.0	4	1.0
The mean is zero	8	5.7	4	2.9	3	2.0	15	3.5
No Response	51	36.5	48	34.3	53	35.3	152	35.4
Others	28	20.0	35	25.0	50	33.4	113	26.1
Total	**140**	**100.0**	**140**	**100.0**	**150**	**100.0**	**430**	**100.0**

From Table 4.24, only between 7.1% and 15.0% across the three year groups were able to indicate that a standard deviation of zero means "all the scores are the same". A larger percentage (10.7% to 15.7%) showed that a zero standard deviation means "poor performance of students". It was also found that between 34.3% and 36.5% of the participants of all year groups did not respond to the item. The remaining gave other responses which included: "the entire students had excellent scores"; "it means the person who calculated the standard deviation cancelled the mean"; "nobody took part in the examination"; etc.

4.4.2 Interview Results on Students' Conceptual Understanding of the Standard Deviation

Adding to the responses to the test items on standard deviation recounted above, interview responses of the nine students who were selected randomly assisted in examining the conceptual understanding of the standard deviation. Excerpts of the interview responses of the participants are presented in the passages below:

Interviewer: Have you heard of the standard deviation before?

Ato: Yes, I learnt it in school.

Interviewer: So, what is it? How will you describe or explain the standard deviation?

Ato: What? Describe? .... (Looked surprised). Is it how to calculate it or....? I only know how it is calculated.

Interviewer: Just tell me everything you know about the standard deviation as a statistical measure.

Ato: Eeeh, the formula is ..... (Looks up and tries to remember the formula). It is x minus the mean, or the other way round?

Interviewer: Yes, go on.

Ato: ..., I think you divide it by ……. And square. In fact, I have forgotten.

Interviewer: You can write it on the paper.

Ato: In fact,I have forgotten. It's like you square a part and find the square root too.

Interviewer: So if you have forgotten the formula, then what will the calculated value represent?

Ato: Madam, I can only say it is the standard deviation.

Interviewer: Doesn't the name sound a bell?

Ato: Errm, Deviation? Away from something ....?

Interviewer: Yes, so how will you link it to the standard deviation?

Ato: (Shakes the head) .... Madam, I was not taught, and I have not read.

Interviewer: Ok, we will continue the discussion in class.

Ato: Thank you, Madam.

No matter how hard the researcher tried to help Ato**,** he could not describe the standard deviation, neither could he even remember the algorithm for finding it. The same situation was encountered during the interaction with six others: Esinam, Mohammed, Laryea, Ebo, Dakpo and Korshi**.** Nonetheless, they demonstrated that when they are given the algorithm, they can do the computation. Exceptionally, Mohammed wrongly added that the standard deviation shows the center of a data set. This was what he had to say:

Mohammed: In fact, I don't remember being taught what the standard deviation is or what it is used for. But I think it is used to show the middle of a set of data.

Interviewer: Middle of data? Are you sure of that?

Mohammed: Hmm … No, (Not quite sure of himself) ….. I think it is the range ….. or the median?

*Interviewer***:** *Do you remember how the standard deviation is calculated?*

Mohammed: It has been some time, so i am not sure I can remember. I did not understand it well at that time. However, if I am given the formula, I can do the computation.

*Interviewer***:** *Oh, Ok, but after the computation, what will the computed value describe about the data set from which it was computed from?*

Mohammed: Hmm. That is what I can tell.

Aside expressing the same view as Ato, Laryea also wrongly added that the standard deviation shows the number of students who failed in a test. This is found in the extract below:

Laryea: The standard deviation is a calculated figure from a set of scores or given data.

Interviewer: Yes, but what does it represent?

Laryea: When it is calculated, it shows the number of students who failed the test. It represents all the set.

Interviewer: How do you mean?

Laryea: When it is zero, then all of the students might have failed the test.

Interviewer: On the other hand?

Laryea: It can show …… Errm, those who did not do well in the test. (Shakes the head)

Interviewer: So if I have a set of scores, for example {5, 10, 13, 20, 23 and 25} how will I find the standard deviation?

Laryea: I think I have forgotten the formula. It's been some time now since I last studied it.

Interviewer: So you don't remember anything at all?

Laryea: I am not sure I do.

Interviewer: Look at these two histograms, (shows the pair of histograms that was in the test items), which of them will have a larger standard deviation?

Laryea: I could not answer this in the test we did yesterday.

Interviewer: Why?

Laryea: We never did something like this in SHS. We were only given a data to calculate mean and standard deviation.

Though Kwesi tried this time again, he could only write the formula for the standard deviation with much difficulty. He also did not show understanding of the concept of standard deviation neither did he demonstrate understanding of the algorithm as shown in the discussion below:

Kwesi: The standard deviation … is it the one that measures the range of a data set? …… (looks in space and scratches the head)

Interviewer: Where the range means? Can you go further?

Kwesi: Umm… is it not the length of the data? (Tries to ask the interviewer)

Interviewer: The length of the data? Can you explain further?

Kwesi: No, but it is like where the data starts from and ends. I am not sure l know what it is, so let me rather write how it is calculated.

Interviewer: Go ahead,

Kwesi: It is x minus the mean, ….. (eehm, yes, so it is a range from the mean – he seems to remember something), then all is squared.

Interviewer: Yes, go ahead.

Kwesi: Then you take the square root,

Interviewer: Go back and check, does it not seem like you have left something out in the formula?

Kwesi: … Umm, …. (Goes over the formula). I think it is divided by the mean again, or?

Interviewer: The mean?

Kwesi: No, (tries to raise the hand and laughs) …. Madam, it is n, that is the number of values in the data. Yes, I remember.

Interviewer: Any idea why the "x minus the mean" is squared?

Kwesi: Ooh, no. That one I don't know. But I think that is why you take the square root again.

Interviewer: So what does the calculated value tell you about the data?

Kwesi: That is the standard deviation.

Interviewer: Any information left to be added?

Kwesi: No, (laughs). I will rather go and find out, since at SHS we were only taught to do the calculation.

Lastly, though Tei tried to explain the algorithm for finding the standard deviation, he demonstrated his understanding as a deviation from the mean instead of average deviation from the mean. This can be found in the extract below:

Tei: It shows …… I think it is the length of the data from one end to the other.

Interviewer: What do you mean by length of data?

Tei: It means you find the difference between the biggest number and the smallest number …. No, that is the range (He remembers). The standard deviation shows the deviation in the data.

Interviewer: Go ahead,

Tei: It is the square root of variance.

Interviewer: So what is variance? And what does it represent?

Tei: I can't tell.

Interviewer: So let's come back to the standard deviation. How will you describe it?

Tei: You find the deviation from the mean and square it. Then… (He writes the correct formula down).

Interviewer: Good. Looking at the formula, anything to be added to what the standard deviation represents?

Tei: No. It is the total deviation from the mean because of the summation sign.

Interviewer: Ok. Imagine two teachers, teacher A and B, taught the same course. On a standardized test, teacher A's class had a standard deviation of 2.4 and on the same test teacher B's class had a standard deviation of 1.2 on the same test. What can you say about the scores of the two classes?

Tei: (writes down the information) …….. Hmm. I will say Teacher B's class did better.

Interviewer: What is your reason?

Tei: *Since Teacher B's class has a smaller standard deviation, it means the arithmetic mean is also larger. Meaning their scores are larger than the scores of Teacher A's class.*

Interviewer: Can you use your definition for the standard deviation to comment?

Tei: *(Smiles and scratches the head). I think I just have to learn hard for the course.*

Interviewer: (Smiles back). Ok.

4.4.3 Discussion of Results on Students' Conceptual Understanding of the Standard Deviation

A number of researchers have found the standard deviation to be a difficult concept for students to comprehend (Shaughnessy, 1997; Reading & Shaughnessy, 2004; Al-Saleh & Yousif, 2009), since most instructors tend to focus on the computations of the concept without helping students develop a conceptual understanding of it (Delmas & Liu, 2005; Inzunza, 2006). According to Inzunza (2006), though studies are now being conducted to investigate students' conception of variability, a limited number is found at the university level, where students must be seen to apply their understanding of the concept to start studying statistical inference.

Results of the current study indicated that fresh undergraduate students could not explain or describe the standard deviation and what it represents. From Table 4.16, only 2.9% of all the three year groups combined were able to describe the standard deviation as either a deviation from the mean or the spread of data. During the interview, none of the interviewees were able to give an appropriate explanation or description of the standard deviation. For Kwame, though the interviewer tried to help him realize that it is the average deviation from the mean, he only saw the standard deviation as a total deviation from the mean because of the summation sign in the algorithm for its computation. Similar to this is Gina in Clark, Kraut, Mathews, and Wimbish's (2007) study, who only recognized the standard deviation as a measure of deviation of data values from the mean, but never as an average deviation from the mean, despite several promptings from the interviewer. This finding is in support of previous studies that have found the standard deviation to be a difficult concept for students to understand (Mathews & Clark, 2003; delMas & Liu, 2005; Dubreil-Frémont, Chevallier-Gaté, & Zendrera, 2014).

As compared to responses on the arithmetic mean, participants in the present study's confidence in responding to the test items on the standard deviation was very low. This is reflected in the high non response rate of the items on the standard deviation. In responding to the item in Box 5 on "what the standard deviation means", an average of 63.6 % from all the three year batches did not respond. Not mentioning those who gave inappropriate responses. This supports the pretest findings of Dubreil-Frémont, Chevallier-Gaté, and Zendrera (2014) in which only 18.5% of participants gave appropriate definitions of the standard deviation. In Dubreil-Frémont, Chevallier-Gaté, and Zendrera's (2014) pretest results, more than 60.0% of the students were unable to define or explain what the standard deviation represented in relation to the data with about 50.0% giving other definitions that had no relation to the concept of the standard deviation.

Surprisingly, when it also came to the computation of the standard deviation in the current study, only 8.2% (of the combined group) was able to do so. There was also a higher non-response rate (47.8%) for the combined group as revealed in Table 4.17. Their inability to compute the standard deviation is contrary to findings that students learn to calculate measures of dispersion like the standard deviation and variance, but do so without understanding its meaning (Inzunza, 2006; Garfield &

Ben-Zvi, 2007). Garfield and Ben-Zvi's (2007) revisiting studies on how students learn statistics also observed that though students may learn how to compute measures of variability, they rarely show understanding of what these summary measures represent, as well as fail to show understanding of their importance and their connectivity to other concepts.

In the same vein, during the interview in the present study, participants could not explain anything about the standard deviation. Unlike being able to describe the algorithm for the arithmetic mean, they could not even state the algorithm, let alone explain the processes involved in the algorithm.

Research question three tried to explore participants' conceptual understanding of the standard deviation. Based on the analysis of participants' responses to test items and interview responses presented above, it can be concluded that first year undergraduate mathematics students do not have any conceptual understanding of the standard deviation. They could not describe or explain what it the standard deviation is; neither could they compute it, let alone to describe the algorithm for its computation.

4.5 Research Question Four - Level of Conceptual Knowledge with Respect to the APOS Framework the Undergraduate Mathematics Students have on the Concept of Standard Deviation.

During the interview, though only Kwesi and Tei could state the algorithm for finding the standard deviation, it was Tei who exhibited some form of an incomplete conception of the algorithm. Nonetheless, like Mohammed, Dakpo also demonstrated the ability to compute the standard deviation when given external promptings. According to the APOS Theoretical framework, an individual whose conception of a mathematical concept is limited to being able to perform actions associated with that concept only when given external promptings is said to have an action conception of the concept (Asiala et al., 1997). This shows that, only a few of the participants have an action conception of the standard deviation. It was further revealed in the analysis of the test results that only 8.2% (average for all year groups) were able to compute the standard deviation of the data in the frequency table provided (Table 4.17). This indicates that only this percentage (8.2%) of participants demonstrated an action conception of the standard deviation.

To exhibit a process conception of the standard deviation, participants must be able to demonstrate an interiorization of the skill of computing the standard deviation. This means participants must be able to recall the algorithm without external prompting, work with it and be able to reverse it. Though Kwesi could recall the algorithm for finding the standard deviation with ease, he did not show any understanding of it. Tei who attempted it could not describe it appropriately. He only saw the standard deviation as the sum of the deviations from the mean as a result of the summation sign in the algorithm. He only demonstrated an instrumental understanding of the algorithm.

From the discussions of students' responses on both the test items and interview responses to the standard deviation, it can be concluded that less than 10.0% of first year undergraduate mathematics students have an action conception of the standard deviation. None of them demonstrated a process conception of the standard deviation. This is consistent with the result of Mathews and Clark (2003), and Clark, Kraut, Mathews, and Wimbish's (2007) study in which only about a third of the participants demonstrated to have the lowest level of conception of the standard deviation, the action conception. Similar to the findings of this study, participants in Mathews and Clark's (2003) study also saw the standard deviation as an algorithm to be given and followed but were unable to describe the algorithm. Mathews and Clark (2003) attributed this to the complicated nature of the algorithm as well as the unjustified use of the algorithm in the minds of the students.

However, contrary to the results of the current study, six students in Clark, Kraut, Mathews, and Wimbish's (2007) study progressed past an action conception and even demonstrated having an object conception of the standard deviation. Just that they differed greatly in the richness of their conceptions. Some demonstrated a relational understanding of the algorithm for computing the standard deviation, while others, an instrumental understanding of this algorithm. According to Clark, Kraut, Mathews, and Wimbish (2007), a student who understands the algorithm of the standard deviation relationally, will consider the successive distances or deviation from the mean of the data set. On the other hand, the student who understands the algorithm instrumentally will just think of it involving subtraction and squaring of numbers. However, in both cases, Clark, Kraut, Mathews, and Wimbish (2007) stated that students would have interiorized the algorithm for finding the standard deviation to a process, and they would be able to think of computing the standard deviation by inputting data, performing computation and getting an output.

From the other test responses, though 55.0% of the participants for all year groups combined was able to select between two data sets which of them had a larger standard deviation (see Table 4.18), their justification for their answer was wrong. Only an average of 13.2% of all participants from the three year groups could give appropriate justifications that either the "data were more spread out" or it had a "higher range" (Table 4.19). Fifty seven of the overall participants, (representing 24.2 %) of those who responded to that item have built an inappropriate conception that if a data has a lower arithmetic mean, then it will have a larger standard deviation. As seen from Table 4.19, there were a number of inappropriate responses with still a high non response rate (22.2%).

Participants found it difficult to select among a pair of histograms, the one with a larger standard deviation. Among an average of 30.9% from all the year groups who were able select the appropriate histogram (Table 4.21), only 5.5% of them were able to give the right justification that "Histogram B was more spread out than Histogram A" (Table 4.23). Similar to the misconception participants demonstrated in the data set, from Table 4.19 that a data set with a larger standard deviation has a lower arithmetic mean; an average of 22.0% of all participants again demonstrated an inappropriate conception that the larger standard deviation of Histogram B is attributed to its lower arithmetic mean (see Table 4.23). The non-response rate, as well as those who answered wrongly was also high for this item, indicating the lack of confidence in responding to the item.

In a study whose purpose was to find out the conception of university students in sampling distributions and related concepts such as variability, Inzunza, (2006) made a similar observation. Though three of the eleven students used were able to identify correctly among two histograms, the one with a larger standard deviation, only two of them were able to give an appropriate justification for their response by considering the wide range of the distribution. Two misconceptions were identified from those who selected the wrong graph. In their justification they stated that: "variability depends on the quantity of data" and "variability also depends on the irregularity of the distribution". These misconceptions also showed up in some listed justifications not tabulated in Table 4.22. Also in Turegun's (2011) study, he observed that only 6.1% of the college students used were able to compare two histograms for the size of their standard deviation.

From Inzunza's (2006) study, it was observed that though students might use correctly some properties of distributions to assign values to the standard deviation, analysis of their responses might reveal isolated and unconnected understanding of the variability in the distribution and the standard deviation assigned to it. It was also observed that students also learn to compute the standard deviation without understanding its meaning when applied in different contexts.

Participants in the present study could not explain the meaning of a zero standard deviation. Only 11.1% (averagely from all year groups) did. Contrary to this result, in Turegun's (2011) study, a higher percentage (42.4%), was able to interpret a zero standard deviation, with 54.5% of them being able to interpret a non-zero standard deviation.

In conclusion, research question four also sought to investigate the level of conceptual knowledge of the standard deviation that first year undergraduate mathematics students have according to the APOS Theory. From participants' responses to test and the interview responses on the standard deviation in the current study, it can be concluded that first year undergraduate mathematics students do not even have an action conception of the standard deviation according to the APOS Theory. Participants were unable to compute the standard deviation of a frequency table without external prompts. They were not able to describe the algorithm; neither could they describe what the computed value represented in relation to the data from which it was computed. They also demonstrated various misconceptions on the concept of the standard deviation.

4.6 Summary

This chapter looked at the analysis and discussion of the qualitative data obtained from both the test and the interview. Explanation of students' responses and reasoning revealed that they have conceptualized the arithmetic mean both as an average and a computational act. As an average, students could not define or explain what an average is. All they know is that the arithmetic mean is an average without knowing what averages are or represents. Only two of the participants, Kwesi and Tei, mentioned the other averages, with Kwesi telling how they can be found. Tei, on the other hand, is the only participant who stated that the averages represent the center of a data set and hence they are called measures of central tendency.

As a computational act, students have conceptualized the arithmetic mean with the "add and divide algorithm without understanding what the outcome of this act represents. When asked, they swing between the arithmetic mean being an average and it being obtained by the "add them up and divide algorithm".

Concerning the level of conceptual knowledge of the arithmetic mean, according to the APOS Theory, first year undergraduate mathematics students demonstrated an incomplete process conception of the arithmetic mean by not being able to reverse the process of computation even though they could describe the process without external prompts. They did not see the arithmetic mean as a concept with properties, neither could they explain the connection between the computed value and the data from which it was computed from.

Students did not demonstrate a conceptual understanding of the standard deviation. They could not define nor explain the concept, neither could they compute it, let alone tell what it represents. They demonstrated that the standard deviation is a difficult concept to learn.

Concerning the level of conceptual knowledge of the standard deviation according to the APOS Theory, the participants in the current study demonstrated that they do not even have an action conception, the lowest level, of the standard deviation. Only one person demonstrated an incomplete process conception of the standard deviation by stating and partially describing the algorithm instrumentally.

CHAPTER FIVE

SUMMARY OF FINDINGS, CONCLUSIONS AND RECOMMENDATIONS

5.0 Overview

This chapter provides a summary of the purpose, methodology, and results of the current study. Conclusions are then discussed based on the insights gained with regard to the findings and limitations of the study. Recommendations are then given for development in teaching and possible areas for further research studies suggested.

5.1 Summary of the Study

The study investigated undergraduate mathematics students' conceptual understanding of the arithmetic mean and the standard deviation at their entry stage into the university based on their mathematical learning experiences at the JHS and SHS It also sought to explore and determine the level of conceptual knowledge of the arithmetic mean and the standard deviation they have with respect to the APOS Theory. The conceptual understanding of the concepts constructed by these students has been done as a result of their mathematical experiences gained in studying core mathematics at the JHS and SHS levels, as well as elective mathematics at the SHS level. In all, four hundred and thirty students from three consecutive year groups of students admitted into the Department of Mathematics Education of the UEW were used as a sample for the study. They comprised, one hundred and forty (140) students, each of the 2012/13 and 2013/14 batches and one hundred and fifty (150) of the 2014/15 batch of students. Two instruments were used for the study: a test and an interview protocol. The test consisted of both objective test items as well as students constructed response items. However, in most of these items participants were to indicate their justification for their responses. For each of the year groups, participants were given the test before formal lectures for the introductory statistics course begun. This was to ensure that there were no interferences in participants' constructed conceptions from their mathematics learning experiences in JHS and SHS core mathematics and SHS elective mathematics. Nine randomly sampled students from the 2014/15 batch were also interviewed about their conceptual understanding of the arithmetic mean and the standard deviation. The interview was conducted a day after the test.

Scoring of the test was done qualitatively. For the objective test items, the researcher considered whether participants were able to get an item correct or wrong, whereas in the constructed response items consideration was given to the common themes that run through the responses. However, in almost all the items, participants had to give justifications for their responses. The percentage of participants whose responses fell under a theme was then noted. The interview also helped the researcher to gain a deeper insight into the participants' conceptual understanding of the concepts under study. The conceptual understanding of the concepts demonstrated by participants in both the test and the interview and the ideas required to be able to answer the test items helped the researcher determine participants' levels of conceptual knowledge according to the APOS theoretical framework.

5.2 Findings

After the analysis of the responses of the test items and interview responses, the following findings were made:

- It was revealed that undergraduate mathematics students have conceptualized the arithmetic mean as an average and also with the algorithmic definition: the "add and divide algorithm". Though they define the arithmetic mean as an average, and the only average they know, students did not demonstrate an understanding of what an average represents. They only used the computational algorithm to define an average without knowing what the outcome of the algorithm represents.
- The conceptual understanding demonstrated by the undergraduate mathematics students in the study indicated that they have an incomplete process conception of the arithmetic mean with respect to the APOS Theoretical Framework. Participants showed mastery of the computational algorithm of the concept without any external promptings. They also demonstrated to have interiorized the algorithm. That is, they demonstrated an understanding of the algorithm and also described it without actually going through the process. However, reversing the algorithm posed a challenge to them. They also did not exhibit an understanding of the properties of the arithmetic mean.
- It was also revealed during the analysis that the undergraduate mathematics students do not have any conceptual understanding of the standard deviation. Participants could not compute the standard deviation, neither could they define or explain what the standard deviation represents. In the same vein,

they did not show understanding of any property of the standard deviation. Rather, it was revealed that about a quarter of them have built a misconception that a data set with a smaller arithmetic mean will have a larger standard deviation, and vice versa.

- This no conceptual understanding of the standard deviation demonstrated by the undergraduate mathematics students indicated that they do not even have an action conception (the lowest level of conceptual knowledge of the APOS framework) of the standard deviation. Only about 8.2% of the participants were able to compute the standard deviation with only 2.9% giving an appropriate definition or explanation to the concept of the standard deviation. As a result, they could not also demonstrate an understanding of the properties of the standard deviation.

5.3 Conclusion

The current study adds to the numerous studies which have observed students' difficulty in understanding the concepts of the arithmetic mean and the standard deviation (Groth, 2006; Mathews & Clark, 2007; Jacobbe, 2008; Dubreil-Frémont, Chevallier-Gaté, & Zendrera, 2014). It is no doubt that both concepts of the arithmetic mean and the standard deviation are two common but very important concepts encountered in any introductory statistics class as a result of their importance and applications to everyday experiences.

In this information age, where people are saturated with data everywhere, it is the objective of every introductory statistics course to help increase student's awareness of data in everyday life and get them ready for future careers (Rumsey, 2002). Students must, therefore, be trained to become statistically literate as research has also revealed that understanding of statistical principles, and their usage appropriately, relate to making quality decisions, judgments and inferences (Kahneman, Slovic, & Tversky, 1982) which are thestatistical reasoning. To be statistically literate, students must be trained to understand and interpret statistical concepts, appraise critically and abstract information from statistical procedures and presentations that are encountered in everyday life. They should also be able to recognize and question assertions that are made without any statistical basis. However, the results from the present study have revealed that much needs to be done to improve on student's learning of the concepts under study.

If the Ghanaian student will be statistically literate and if the objective of helping the Ghanaian child to be able to collect, analyze and interpret data and find probability of events as spelt out in the Ghanaian primary mathematics curriculum is to be achieved, then based on the results of this study, measures must be put in place to help students develop conceptual understanding of the concepts of the arithmetic mean and the standard deviation which serve as foundational concepts for other statistical concepts applied in many fields.

5.4 Recommendations

From the findings and conclusion above, the following recommendations are made:

- That lecturers and tutors in Ghanaian teacher training institutions must ensure they teach teacher trainees the concepts of the arithmetic mean and the standard deviation before teaching them any computations on these concepts.

- That curriculum developers review the JHS and SHS mathematics syllabi to include the teaching of statistical concepts before computations are done on these concepts.
- That the Ghana Education Service (GES) and the Mathematics Association of Ghana (MAG) must help train and task teachers to ensure that the teaching of statistics becomes real or practical.
- That the Ghana Education Service (GES) and the Mathematics Association of Ghana (MAG) again train mathematics teachers at both the JH. and SHS levels on how to set questions that will encourage statistical thinking and reasoning rather than those which require just the recall of facts.

In the mathematics syllabus for JHS and SHS, the specific objective for teaching the arithmetic mean is that teachers are to teach students to calculate the arithmetic mean using appropriate formula (CRDD, 2010; CRDD, 2012b). The JHS and SHS syllabi must be ammended in such a way that the concepts of the arithmetic mean and the standard deviation will be presented or taught to students for them to know what they are and what they represent. By this students will be taught the properties of these concepts before doing computations on these concepts.

Real life situations or problem-solving situations must be included or presented during teaching sessions so that students will be taught how to use the knowledge gained to solve everyday problems and also apply them in different situations. This will help eradicate the conventional way of decontextualizing the teaching of mathematics in Ghana as observed by Kwarteng and Ahia (2004).

If teachers are trained on setting questions that will require students to use their statistical reasoning and thinking skills. This will help students to demonstrate their conceptual understanding of concepts and in so doing those with difficulties can easily be identified and assisted. All these will enhance students' conceptual understanding of the concepts of the arithmetic mean and the standard deviation to help them become statistical literates. Then they can use their statistical literacy to reason as well as think statistically.

5.5 Areas for Further Research

Based on the outlined limitations, this study becomes a useful baseline study for future research in the Ghanaian setting in the following areas:

- replicating the study to include students from other departments and universities who studied elective mathematics at the SHS level and the results compared with that of the current study.
- conducting a similar study to explore SHS mathematics teachers conceptual understanding on the concept of the arithmetic mean and the standard deviation.

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APPENDICES

Appendix A **Letter for Consent**

You are welcome to the University of Education, Winneba and as a student of the Department of Mathematics Education. We hope you are going to enjoy your studies as you put in your maximum best. All the lecturers in the department are ready to help you to achieve your goals for pursuing this program.

To start this course, Probability and Statistics I, I would like to know your background in some of the basic concepts we are going to study by giving you a test. Results obtained will be used in a study to investigate students' conceptual knowledge of the arithmetic mean and the standard deviation. Consideration will be given to resulting implications according to their effect on mathematics education and recommended measures made to improve on the effectiveness of the teaching and learning of these concepts at the basic and senior high levels.

Your background in senior high core and elective mathematics is enough as a prerequisite knowledge for this test. Since the results from this test is not going to be part of your assessment, feel free to answer all questions as frankly as possible. Your participation is very important, however, you have the right to decline to participate in the study.

If you agree to be part of this study, kindly give your consent by filling the consent form attached. Thank you for your consideration.

…………………………

Armah, Gloria (Mrs.)

Appendix B Participant's Consent Form

I, _______________________________, a first year student of the Department of Mathematics Education of the University of Education, Winneba, give my consent to be part of this study. I understand the study will involve completing questionnaire, a test and an interview. I understand that all information including my student's identification number will be kept confidential. I understand that these activities will not disrupt my programme and results of the test will not form part of my assessment.

Signed: ……………………………………….

Date: ………………………………………….

Appendix C Test for Students

UNIVERSITY OF EDUCATION, WINNEBA
FACULTY OF SCIENCE EDUCATION
DEPARTMENT OF MATHEMATICS EDUCATION

B. Sc. (MATHEMATICS EDUCATION) 2014/2015
MATD 113 (PROBABILITY AND STATISTICS I)

SEPT. 24, 2014

TEST FOR LEVEL 100 STUDENTS

PART A: PERSONAL DATA

INDEX NO. (LAST THREE DIGITS ONLY): ………………

INSTRUCTIONS: Answer all questions. Tick as appropriate, and where possible provide short answers.

1. Gender: Male [ ] Female [ ]

2. Age: Below 20 [ ] 20 – 24 [ ]
   25 – 29 [ ] 30 – 34 [ ]
   35 – 39 [ ] 40 and above [ ]

3. What program did you study at the S. H. S level? ……………..……………………

4. In which year did you complete S. H. S? ………………………...…………

5. In which region did you attend S. H. S? …………………………………….

6. Have you studied any advanced mathematics after S. H. S? Yes [ ]No [ ]

7. Have you taught before? Yes [ ] No [ ]

8. Are you a trained teacher? Yes [ ] No [ ]

9. If your answer to 7 is yes, then at what level of education did you teach?

Pre – school Level [ ] Primary Level [ ]

J. H. S. Level [ ] S. H. S. Level [ ]

a. Did you teach mathematics? Yes [ ] No [ ]

10. Did you study Elective Mathematics in S. H. S.? Yes [ ] No [ ]

PART B: TEST ON ARITHMETIC MEAN AND STANDARD DEVIATION

The table below represents the distribution of ages of a group of students taking a course. Use it to answer questions 11to 13.

Age (x)	5	6	7	8	9
Freq. (N)	2	3	3	2	1

11. A student said the arithmetic mean of the data is 10. Without calculating can this be true? Why or why not?

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12. Find the arithmetic mean for this distribution.

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…...…………………………………………………………………………………

13. Find the standard deviation for this data?

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14. Kwesi and Adwoa are analyzing the following data sets.

Data Set A: 110, 112, 114, 115, 116, 118.

Data Set B: 2, 6, 15, 28, 59, 112.

a. Without calculating, which data set, A or B, would have the larger arithmetic mean? Explain your answer.

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b. Without calculating, which of the data sets, A or B will have the larger standard deviation? Why?

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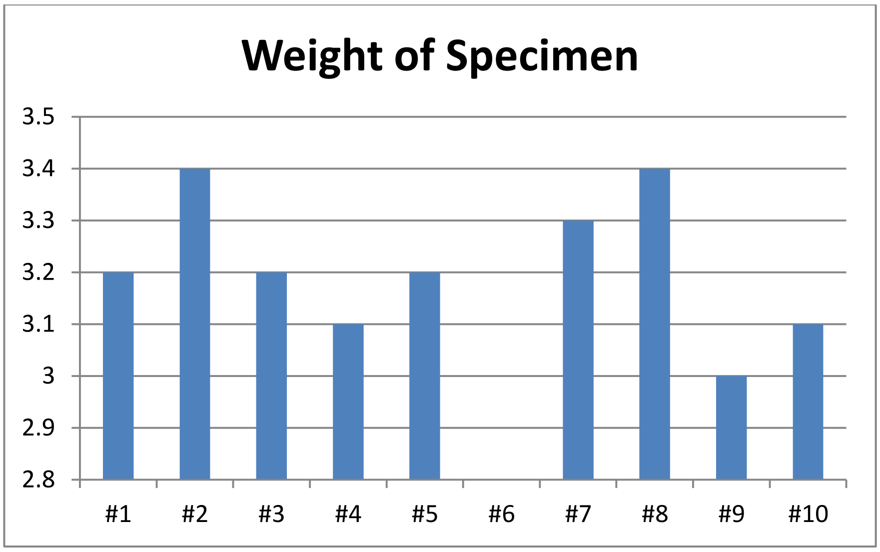
15. In a chemistry lab a student weighed a specimen ten times. The results of those weighings are presented in the bar chart below. The student lost the 6th weighing of the specimen after she calculated the mean of the ten weighings to be 3.2 as indicated by the dark line in the graph below. What could have been the value for the 6th weighing if the mean is 3.2?



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16. A class of 160 sociology students took a final exam, and their arithmetic mean was found to be 125.6. What does it mean to say that the arithmetic mean of all the final exam scores is 125.6?

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17. What is the purpose of finding the arithmetic mean of a data set?

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18. Imagine a set of quiz scores. If you were told that the standard deviation of the quiz scores is 0, what would that tell you about the scores?

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19. Here are values of measurements of distances (in mm) between the two eyes of 8 adult patients who were to be given eyeglasses: 67, 66, 59, 62, 63, 66, 55, and 66. The mean distance is 63 mm and the standard deviation is 4.2 mm. Explain to someone not in a statistics class, what the standard deviation means.

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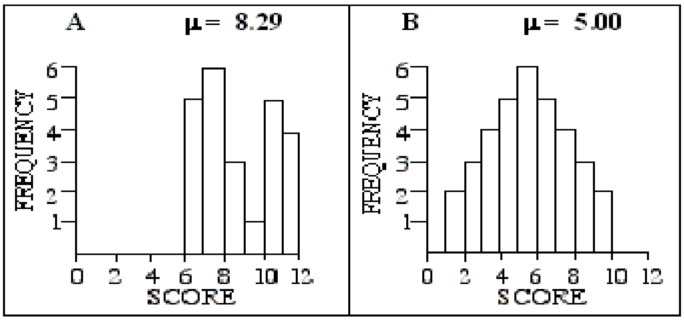
20. For the pair of histogram below, determine which histogram has the larger standard deviation (it is not necessary to do any calculations to answer this question).



A. A has a larger standard deviation than B

B. B has a larger standard deviation than A

C. Both graphs have the same standard deviation

Explain your answer.

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Appendix D **Student's Interview Guide**

INDEX NO. (LAST THREE DIGITS ONLY) ……………… Date: …….

1. Gender: Male [ ] Female [ ]

2. Age: Below 20 [ ] 20 – 24 [ ]

   25 – 29 [ ] 30 – 34 [ ]

   26 – 39 [ ] 40 and Above [ ]

3. What program did you study at the S. H. S level? ………………………

4. In which year did you complete S. H. S? ………………..………...…

5. Have you studied any advanced mathematics after S. H. S?

   Yes [ ] No [ ]

6. Did you study Elective Mathematics at SHS? Yes [ ] No [ ]

7. What is meant by the term "arithmetic mean" in statistics?

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   (Probe further depending on response given)

8. What is meant by the term "standard deviation" in statistics? ………………

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   (Probe further depending on response given)

Appendix E **Sample of Student's Responses on the Test**

UNIVERSITY OF EDUCATION, WINNEBA
FACULTY OF SCIENCE EDUCATION
DEPARTMENT OF MATHEMATICS EDUCATION

B. Sc. (MATHEMATICS EDUCATION) 2014/2015
MATD 113 (PROBABILITY AND STATISTICS I)

SEPT. 24, 2014

TEST FOR LEVEL 100 STUDENTS

PART A: PERSONAL DATA

INDEX NO. (LAST THREE DIGITS ONLY): 072

INSTRUCTIONS: Answer all questions. Tick as appropriate, and where possible provide short answers.

1. Gender: Male [✓] Female [ ]
2. Age: Below 20 [ ] 20 – 24 [ ]
   25 – 29 [✓] 30 – 34 [ ]
   35 – 39 [ ] 40 and above [ ]
3. What program did you study at the S. H. S level? BUSINES
4. In which year did you complete S. H. S? 2012
5. In which region did you attend S. H. S? Brong Ahafo Region
6. Have you studied any advanced mathematics after S. H. S? Yes [ ] No [✓]
7. Have you taught before? Yes [ ] No [ ]
8. Are you a trained teacher? Yes [ ] No [✓]
9. If your answer to 7 is yes, then at what level of education did you teach?
   Pre – school Level [ ] Primary Level [ ]
   J. H. S. Level [ ] S. H. S. Level [ ]

a. Did you teach mathematics? Yes [ ] No [✓]

10. Did you study Elective Mathematics in S. H. S.? Yes [✓] No [ ]

PART B: TEST ON ARITHMETIC MEAN AND STANDARD DEVIATION

The table below represents the distribution of ages of a group of students taking a course. Use it to answer questions 11.to 13.

Age (x)	5	6	7	8	9
Freq. (N)	2	3	3	2	1

11. A student said the arithmetic mean of the data is 10. Without calculating can this be true? Why or why not?

yes because the mean determines the deviation

12. Find the arithmetic mean of this distribution.

Mean $(\bar{x}) = \frac{\sum fx}{\sum f} = \frac{35}{11}$

13. Find the standard deviation of this data?

Mean deviation $= 1 - \frac{\sum fx}{\sum f}$

$= 1 - \frac{35}{11}$

$= -\frac{24}{11}$

14. Kwesi and Adwoa are analyzing the following data sets.

Data Set A: 110, 112, 114, 115, 116, 118.

Data Set B: 2, 6, 15, 28, 59, 112.

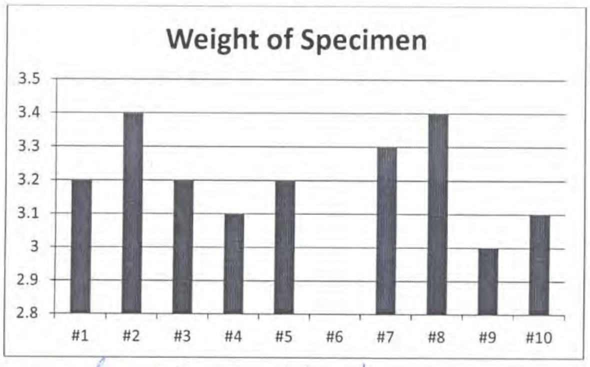
a. Without calculating, which data set, A or B, would have the larger arithmetic mean? Explain your answer.

A because the number are large but the number of occurence are the same

b. Without calculating, which of the data sets, A or B will have the larger standard deviation? Why?

B will have the Large standard deviation because the mean is large.

15. In a chemistry lab a student weighed a specimen ten times. The results of those weighings are presented in the bar chart below. The student lost the 6th weighing of the specimen after she calculated the mean of the ten weighings to be 3.2 as indicated by the dark line in the graph below. What could have been the value for the 6th weighing if the mean is 3.2?



Let the 6th weight be x

$x + \frac{129.2}{6} = 3.2$

$6x + 129.2 = 19.2$

$x = \frac{19.2}{6} - \frac{129.2}{6}$

$x + \frac{129}{16} = 3.2$

16. A class of 160 sociology students took a final exam, and their arithmetic mean was found to be 125.6. What does it mean to say that the arithmetic mean of all the final exam scores is 125.6?

Because the 125.6 shows the average Scores of the student who took the examination

17. What is the purpose of finding the arithmetic mean of a data set?

To know the average

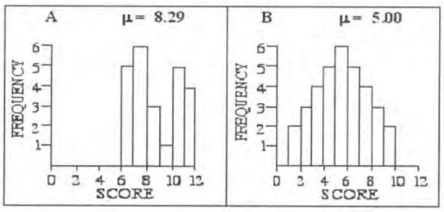
18. Imagine a set of quiz scores. If you were told that the standard deviation of the quiz scores is 0, what would that tell you about the scores?

It means no body deviated or fail.

19. Here are values of measurements of distances (in mm) between the two eyes of 8 adult patients who were to be given eyeglasses: 67, 66, 59, 62, 63, 66, 55, and 66. The mean distance is 63 mm and the standard deviation is 4.2 mm. Explain to someone not in a statistics class, what the standard deviation means.

It means that 4.2mm is not fitted for the eye glasses or 4.2mm does not fall within the distance fitted for the eyeglasses

20. For the pair of histogram below, determine which histogram has the larger standard deviation (it is not necessary to do any calculations to answer this question).



A. A has a larger standard deviation than B

B. B has a larger standard deviation than A

C. Both graphs have the same standard deviation

Explain your answer.

Since the mean for set B is less it means its standard deviation will be higher

UNIVERSITY OF EDUCATION, WINNEBA
FACULTY OF SCIENCE EDUCATION
DEPARTMENT OF MATHEMATICS EDUCATION

B. Sc. (MATHEMATICS EDUCATION) 2012/2013
MATD 113 (PROBABILITY AND STATISTICS I)

SEPT. 17, 2012

TEST FOR LEVEL 100 STUDENTS

PART A: PERSONAL DATA

INDEX NO. (LAST THREE DIGITS ONLY): ...022...

INSTRUCTIONS: Answer all questions. Tick as appropriate, and where possible provide short answers.

1. Gender: Male [ ] Female [✓]
2. Age: Below 20 [✓] 20 – 24 [ ]
   25 – 29 [ ] 30 – 34 [ ]
   35 – 39 [ ] 40 and above [ ]
3. What program did you study at the S. H. S level? ...General Science...
4. In which year did you complete S. H. S? ...2012...
5. In which region did you attend S. H. S? ...Upper West...
6. Have you studied any advanced mathematics after S. H. S? Yes [ ] No [✓]
7. Have you taught before? Yes [ ] No [ ]
8. Are you a trained teacher? Yes [ ] No [✓]
9. If your answer to 7 is yes, then at what level of education did you teach?
   Pre – school Level [ ] Primary Level [ ]
   J. H. S. Level [ ] S. H. S. Level [ ]

a. Did you teach mathematics? Yes [ ] No [ ]

10. Did you study Elective Mathematics in S. H. S.? Yes [✓] No [ ]

PART B: TEST ON ARITHMETIC MEAN AND STANDARD DEVIATION

The table below represents the distribution of ages of a group of students taking a course. Use it to answer questions 11.to 13.

Age (x)	5	6	7	8	9
Freq. (N)	2	3	3	2	1

11. A student said the arithmetic mean of the data is 10. Without calculating can this be true? Why or why not?

Not true

12. Find the arithmetic mean of this distribution.

$\text{mean} = \frac{\Sigma fx}{\Sigma f} = \frac{74}{11} = 6.727$

13. Find the standard deviation of this data?

standard deviation =

14. Kwesi and Adwoa are analyzing the following data sets.

Data Set A: 110, 112, 114, 115, 116, 118.

Data Set B: 2, 6, 15, 28, 59, 112.

a. Without calculating, which data set, A or B, would have the larger arithmetic mean? Explain your answer.

Set A the because when the frequence of data A increase, the mean will also increase

b. Without calculating, which of the data sets, A or B will have the larger standard deviation? Why?

B because, when the average of a frequence is high, the

15. In a chemistry lab a student weighed a specimen ten times. The results of those weighings are presented in the bar chart below. The student lost the 6th weighing of the specimen after she calculated the mean of the ten weighings to be 3.2 as indicated by the dark line in the graph below. What could have been the value for the 6th weighing if the mean is 3.2?

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16. A class of 160 sociology students took a final exam, and their arithmetic mean was found to be 125.6. What does it mean to say that the arithmetic mean of all the final exam scores is 125.6?

It means some of the students had heigher marks thats is the

Students scores were higher

17. What is the purpose of finding the arithmetic mean of a data set?

To know the average of the data set

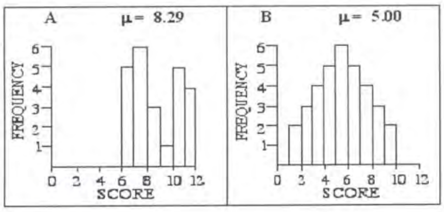
18. Imagine a set of quiz scores. If you were told that the standard deviation of the quiz scores is 0, what would that tell you about the scores?

It means the number zero of people who took part in the quiz scored zero

19. Here are values of measurements of distances (in mm) between the two eyes of 8 adult patients who were to be given eyeglasses: 67, 66, 59, 62, 63, 66, 55, and 66. The mean distance is 63 mm and the standard deviation is 4.2 mm. Explain to someone not in a statistics class, what the standard deviation means.

The standard deviation is the sum up of the mean which is been subtracted ~~and~~ from the value of measurements of distint and is then divided by the sum of number of pupil involved.

20. For the pair of histogram below, determine which histogram has the larger standard deviation (it is not necessary to do any calculations to answer this question).



A. A has a larger standard deviation than B

B. B has a larger standard deviation than A

C. Both graphs have the same standard deviation

Explain your answer.

The mean of B is larger than A, the biger the mean the large ~~the~~ standard deviation